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A Reduced Tensor Product of Braided Fusion Categories over a Symmetric Fusion Category

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Abstract

The main goal of this thesis is to construct a tensor product on the 2-category $\mathbf{BFC}/\mathcal{A}$ of braided fusion categories containing a symmetric fusion category $\mathcal{A}$. We achieve this by introducing the new notion of $\mathcal{Z}(\mathcal{A})$-crossed braided categories. These are categories enriched over the Drinfeld centre $\mathcal{Z}(\mathcal{A})$ of the symmetric fusion category. We show that $\mathcal{Z}(\mathcal{A})$ admits an additional symmetric tensor structure $\otimes_s$, which makes it into a 2-fold monoidal category. By Tannaka duality, $\mathcal{A} \cong \mathrm{Rep}(G)$ (or $\mathrm{Rep}(G,\omega)$) for a finite group G (or finite super-group G). Under this identification $\mathcal{Z}(\mathcal{A}) \cong \mathbf{Vect}_G[G]$, the category of G-equivariant vector bundles over G, and we show that the symmetric tensor product corresponds to (a super version of) to the fibrewise tensor product. We use the additional symmetric tensor product on $\mathcal{Z}(\mathcal{A})$ to define the composition in $\mathcal{Z}(\mathcal{A})$-crossed braided categories, whereas the usual tensor product is used for the monoidal structure. We further require this monoidal structure to be braided for the switch map that uses the braiding in $\mathcal{Z}(\mathcal{A})$.

We show that the 2-category $\mathcal{Z}(\mathcal{A})$-$\mathbf{XBF}$ is equivalent to both $\mathbf{BFC}/\mathcal{A}$ and the 2-category of (super)-G-crossed braided categories. Using the former equivalence, the reduced tensor product on $\mathbf{BFC}/\mathcal{A}$ is defined in terms of the enriched Cartesian product of $(\mathcal{Z}(\mathcal{A}), \otimes_s)$-enriched categories on $\mathcal{Z}(\mathcal{A})$-$\mathbf{XBF}$.

The reduced tensor product obtained in this way has as unit $\mathcal{Z}(\mathcal{A})$. It induces a pairing between minimal modular extensions of categories having $\mathcal{A}$ as their Müger centre.

To my father, Jos Wasserman, whom I miss dearly.

Contents

Chapter 1

Introduction

This thesis concerns a construction of a tensor product of categories for braided fusion categories that contain a fixed full symmetric subcategory $\mathcal{A}$. The construction goes through an enrichment of the braided fusion category over the Drinfeld centre $\mathcal{Z}(\mathcal{A})$ of the symmetric fusion category $\mathcal{A}$. To facilitate this construction, we define an additional monoidal structure on the Drinfeld centre that is laxly compatible with its usual monoidal structure. We examine what the various ingredients of this construction look like from the point of view of Tannaka duality for $\mathcal{A}$. We will always work over the field of complex numbers.

1.1 Motivation and background

The construction done in this thesis is motivated by three closely related questions and existing constructions. An overview of related literature is given in Section 1.4.

1.1.1 Braided Fusion Categories

Braided fusion categories have been extensively studied in the past twenty years. Recall that a fusion category is a linear category (it is enriched and tensored over the category of complex vector spaces) that is tensor (comes equipped with monoidal structure that factors over the tensor product of vector spaces on the hom-objects), semi-simple with finitely many isomorphism classes of simple objects and rigid (every object has left and right duals). A fusion category is called braided if it comes equipped with an isomorphism between $a \otimes b$ and $b \otimes a$ that is natural in a and b and satisfies a compatibility with the associators for the tensor product. Such braided fusion categories are of independent interest as the simplest examples of braided monoidal categories, and some progress has been made towards a classification. As we will describe below, (braided) fusion categories are closely related to topological systems in condensed matter physics and to topological field theories in low dimensions. We will focus our discussion here on the case of ribbon fusion categories (these are braided fusion categories that come equipped with a monoidal natural isomorphism between the identity functor and the double dual functor that is compatible with the braiding). The results in this thesis only require the symmetric fusion category $\mathcal{A}$ to be ribbon.

Modular Tensor Categories

Modular tensor categories are examples of ribbon fusion categories for which the braiding is maximally non-symmetric. Over the complex numbers, a ribbon fusion category $\mathcal{M}$ is modular if and only if its Müger centre or full subcategory of transparent subobjects $\mathcal{Z}_2(\mathcal{M})$, satisfies

$$\mathcal{Z}_2(\mathcal{M}) = \mathbf{Vect},$$

where an object is considered transparent if its double braiding with any other object is trivial. This property is also called non-degeneracy.

Modular tensor categories are of particular interest because they classify once-extended oriented (signature extended) three-dimensional topological quantum field theories with values in the bicategory of linear categories. This classification result is especially appealing because it provides a bridge between algebra and topology, one can directly link the structures a modular tensor category has (such as the monoidal structure and the braiding) to generators of the once-extended three-dimensional bordism category, and link the properties these structures have to relations between these generators.

Minimal Modular Extensions

Not every ribbon fusion category is modular, and, over the complex numbers, the Müger centre measures the failure of a ribbon fusion category $\mathcal{C}$ to be modular. The Müger centre $\mathcal{Z}_2(\mathcal{C})$ is always a symmetric fusion category $\mathcal{A}$. One can, provided that $\mathcal{A}$ is Tannakian (admits a braided functor to **Vect**), find a modular tensor category called modularisation of $\mathcal{C}$ by essentially taking the quotient over $\mathcal{A}$. Several constructions of this modularisation exist, we will discuss them below.

As an alternative way to produce a modular tensor category out of a ribbon fusion category, one can attempt to find a so-called minimal modular extension of $\mathcal{C}$. This is a modular tensor category $\mathcal{M}$ that contains $\mathcal{C}$ as a braided subcategory, with the minimality condition that the full subcategory $\mathcal{Z}_2(\mathcal{C}, \mathcal{M})$ of $\mathcal{M}$ of objects transparent with respect to the objects of $\mathcal{C}$ satisfies $\mathcal{Z}_2(\mathcal{C}, \mathcal{M}) = \mathcal{A}$. One can think of a minimal modular extension as the result of a process where one adds objects to $\mathcal{C}$ that braid non-trivially with the objects of $\mathcal{A}$. The minimality condition then says that no new objects have been added that braid trivially with $\mathcal{A}$.

Minimal modular extensions do not always exist for a given ribbon fusion category $\mathcal{C}$. However, if the set of minimal modular extensions of $\mathcal{C}$ is non-empty, it is a torsor for the minimal modular extensions of $\mathcal{Z}_2(\mathcal{C})$. The reduced tensor product defined in this thesis specialises to the action of the minimal modular extensions of $\mathcal{Z}_2(\mathcal{C})$ on the minimal modular extensions of $\mathcal{C}$.

In condensed matter physics, finding minimal modular extensions can be viewed as a form of gauge fixing, this is discussed below.

1.1.2 G-equivariant Topological Quantum Field Theory

The definition of the reduced tensor product given in this thesis is inspired by considerations about topological quantum field theories. We will outline these here. Recall that a topological quantum field theory is a symmetric monoidal functor out of a bordism category (with objects closed manifolds of a fixed dimension, morphisms between such objects manifolds with boundary that have the disjoint union of the objects as their boundary) into the category of (super-)vector spaces. We can consider bordism categories for different tangential structures, such as orientation, spin structure or, as we will focus on here, oriented with principal G bundles for a finite group G. We will refer to field theories on the latter bordism category as G-equivariant field theories. Furthermore, using the language of higher categories, where one allows $(k+1)$-morphisms between k-morphisms, one can speak about extended topological quantum field theories. Here, one takes the morphisms between the morphisms of the bordism categories to be bordisms between the bordisms defining the morphisms. On the target side, one then needs to replace the category of (super-)vector spaces by a categorification thereof. By a categorification one understands a higher symmetric monoidal category such that repeatedly taking endomorphisms of the monoidal unit and its identity morphisms yields the original category. A field theory that has as objects 0-dimensional manifolds is called fully extended. The dimension of the highest dimensional manifolds that appear in the bordism category for a field theory is referred to as the dimension of that field theory. A field theory on a k-dimensional bordism category where the objects are closed $(k-n)$-manifolds are called n-fold extended.

Dijkgraaf-Witten Theory

The simplest examples of topological quantum field theories that illustrate our considerations are Dijkgraaf-Witten theories. Dijkgraaf-Witten theories is the name given to a family of oriented topological quantum field theories that can be viewed as a fully extended theories and exist in every dimension. Dijkgraaf-Witten theories are sometimes called finite gauge theories, to emphasise their close relation to gauge theories in Physics. In its simplest form the input for a Dijkgraaf-Witten theory is a finite group G. It is usually defined in two steps. The first is to view the assignment $M \mapsto \mathbf{Bun}_G(M)$, that assigns to a manifold the groupoid of principal fibre bundles with fibre G on that manifold, as a functor from an oriented (n-fold extended) bordism category to the category of (n-fold) spans in groupoids. The second step is a linearisation step, in which one assigns to each object in spans of groupoids bundles of (a categorification of) vector spaces.

As an example, if we consider the once-extended three-dimensional oriented Dijkgraaf-Witten, we can examine what it assigns to a circle. We will give two constructions of this value. For the first, observe that the groupoid of principal G-bundles on the circle is the action groupoid for G acting on itself by conjugation. Vector bundles on this are G-equivariant vector bundles over G. The pair of pants morphism from two circles to one equips this linear category with its convolution tensor product. The mapping cylinder for the diffeomorphism that interchanges the legs on the pair of pants gives a braiding for this tensor product. The braided monoidal category found in this way is the Drinfeld centre of the representation category $\mathrm{Rep}(G)$ of the finite group.

We remind the reader that the Drinfeld centre of a monoidal category $(\mathcal{V}, \cdot)$ has objects pairs (z, β), where z is an object of the monoidal category and the half-braiding β is a monoidal natural isomorphism between $- \cdot z$ and $z \cdot -$. This a monoidal category, with monoidal structure $(z, \beta) \otimes_c (z', \beta') = (z \cdot z', (\beta' \cdot \mathrm{id}_{z'}) \circ (\mathrm{id}_z \cdot \beta'))$. Braiding along the half-braidings makes this into a braided monoidal category. As morphisms, we use those morphisms in $\mathcal{V}$ that commute with the half-braidings.

Another way in which one can obtain the same Dijkgraaf-Witten theory is by starting with a topological quantum field theory on the three-dimensional once-extended oriented bordism category with principal G-bundles, denoted $\mathbf{Bord}^{G,\mathrm{or}}_{1,2,3}$. The circle then in particular admits a principal G-bundle (not taken up to equivalence) for each element g of the group, to each of these our topological quantum field theory assigns **Vect** viewed as an object of the 2-category of linear categories. We can view this data altogether as a G-graded linear category $\mathcal{DW}$ with **Vect** in every degree. The pair of pants equipped with principal G-bundles will give rise to a graded tensor product on $\mathcal{DW}$, while the mapping cylinders for the deck transformations will give an action of G on $\mathcal{DW}$ that conjugates the grading. This category is braided up to this G-action. This structure makes $\mathcal{DW}$ into a so-called G-crossed braided category. To obtain an oriented theory from this G-equivariant theory, we can take the homotopy fixed points for the action of G on $\mathcal{DW}$, the category of homotopy fixed points will again be the Drinfeld centre of $\mathrm{Rep}(G)$.

The second construction of Dijkgraaf-Witten theory can be modified using a cocycle in $\alpha \in H^3(G, \mathbf{U}(1))$. In the second description of Dijkgraaf-Witten we can use this to give a non-trivial associator for the graded tensor product on $\mathcal{DW}$, we will denote the resulting category by $\mathcal{DW}(\alpha)$. In this way one obtains a $H^3(G, \mathbf{U}(1))$'s worth of non-equivalent oriented theories, that assign the Drinfeld centre of G-graded vector spaces with associator defined by the cocycle to the circle.

Any category of functors into a symmetric monoidal category carries a monoidal structure, given by multiplying the values two functors take on objects and morphism. In this way, the category of topological quantum field theories on a given bordism category into a given target category carries a monoidal structure. For once-extended three dimensional oriented theories with values in the category of linear categories **LinCat**, this monoidal structure is given by the Deligne tensor product (this is the **Vect**-enriched Cartesian product). Taking the product of a Dijkgraaf-Witten theory with trivial cocycle with itself, one obtains the Dijkgraaf-Witten theory for $G \times G$, with trivial cocycle. However, for field theories on $\mathbf{Bord}_{1,2,3}^{G;\mathrm{or}}$, the tensor product of field theories becomes the degree-wise Deligne product of G-graded categories. In particular, as **Vect** is the monoidal unit in **LinCat**, the category $\mathcal{DW}$ with trivial associator will be the monoidal unit in the category of field theories. Furthermore, this product of G-equivariant field theories makes the $\mathcal{DW}(\alpha)$ into an abelian group which is isomorphic to $H^3(G, \mathbf{U}(1))$. Conversely, one can show that all invertible G-equivariant theories are in this abelian group.

The reduced tensor product described in this thesis is based on the idea that, if one knows an oriented field theory comes from a G-equivariant field theory, there should be a corresponding product of categories that takes this origin into account, like the degreewise tensor product does.

From G-equivariant to oriented

The homotopy fixed point construction used in the discussion of Dijkgraaf-Witten should work more generally. There is evidence that G-equivariant oriented once-extended three-dimensional topological quantum field theories are in one-to-one correspondence with so-called G-crossed modular tensor categories (particular G-crossed braided categories). These are, in turn, in one-to-one correspondence with modular tensor categories containing $\mathrm{Rep}(G)$ as a full symmetric subcategory, where getting from G-crossed modular categories to modular tensor categories is done by finding the homotopy fixed points.

At the level of topological quantum field theories, this taking homotopy fixed points should correspond to passing from G-equivariant theories to oriented theories. From this we infer that there should be a tensor product of modular tensor categories containing $\mathrm{Rep}(G)$ that corresponds to the degreewise product of G-crossed braided categories. The reduced tensor product will be this product.

1.1.3 Short Range Entangled Phases

Topological quantum field theories are intimately related with certain types of condensed matter systems. In particular, there is a close relation between G-equivariant theories and so-called short range entangled phases.

Fusion categories in Physics

In theoretical condensed matter physics, one is led to consider physical systems in which excitations behave like particles. These excitations are then called quasi-particles. In particular kinds of systems, known as topological phases of matter, observables only depend on topological properties of the system. In two spatial dimensions, there are such systems of quasi-particles where observables only depend on the topology of the path the quasi-particles take, that is, on how they wind around each other, and on whether they merge. One can describe such systems in terms of braided fusion categories. Quasi-particles correspond to objects, viewing the monoidal unit as the trivial excitation, the braiding corresponds to the winding, while the fusion corresponds to the merging of the quasi-particles.

From this point of view, non-degeneracy of a braided tensor category means that all quasi-particles apart from the ground state (trivial excitation) can be measured by winding the quasi-particles around each other.

Minimal Modular Extension as Gauge Fixing

Suppose now that we have a degenerate ribbon fusion category, that is, we assume its Müger centre is non-trivial. Viewing the objects in this category as quasi-particles as described above, we can interpret the Müger centre as unobservable degrees of freedom. Unobservable degrees of freedom signify gauge freedom, and, because the Müger centre is symmetric, by Tannaka duality (discussed below) this freedom will be an action of a finite group. Such systems are also called short-range entangled phases.

When faced with gauge freedom, one wants to fix a gauge, which is often done by adding so-called ghost particles to the system that enforce constraints on the system. Reasoning by analogy, we should add ghost particles that make the transparent objects observable, but do not interact any further with the physical excitations. This leads to the idea that gauge fixing for the physical system described by the degenerate ribbon fusion category is done by finding a minimal modular extension for this category.

Combining two such systems corresponds to taking a tensor product of ribbon fusion categories. Here, care should be taken to tensor the systems remembering the gauge freedom. Just as in the case of oriented topological quantum field theories obtained from a G-equivariant theory, taking the Deligne tensor product does not incorporate this. The Deligne tensor product of two minimal modular extensions corresponds to a gauge fixed theory with gauge group $G \times G$. The appropriate tensor product is again the one that comes from remembering that we are dealing with theories that come from G-equivariant theories. The reduced tensor product defined in this thesis corresponds to taking the tensor product of systems with gauge freedom in this way.

1.1.4 Finite groups and Fusion Categories

As we have seen above, there is a strong connection between braided fusion categories containing a symmetric fusion category and G-crossed braided categories for a particular group G. This correspondence comes about in two steps. The first is Tannaka duality for symmetric fusion categories, which tells us symmetric fusion categories are representation categories of finite groups. The second is a pair of mutually inverse constructions known as equivariantisation and de-equivariantisation.

Tannaka Duality for Symmetric Fusion Categories

The relationship between symmetric fusion categories and finite groups is captured by Tannaka duality. The form of Tannaka duality we will be using in this thesis is due to Deligne. His Theorem roughly says that every symmetric fusion category is braided monoidally equivalent to the representation category of a finite (super-)group. The notion of finite super-group used here is that of a finite group equipped with a choice of central element of order two.

Part of the content of this Theorem is that every symmetric fusion category admits a braided monoidal functor to the category of super-vector spaces, called the fibre functor. If the essential image of this functor is contained in the subcategory of vector spaces, the symmetric fusion category is called Tannakian, otherwise we will call it super-Tannakian. The finite (super-)group is found by computing the (super-)group of tensor automorphisms of the fibre functor. In the super-Tannakian case, the grading involution of super-vector spaces will always be an automorphism of the fibre functor and gives rise to the central order two element of the super-group.

(De)-Equivariantisation

There are two existing mutually inverse constructions to pass between braided fusion categories containing a symmetric fusion category and G-crossed braided fusion categories. These constructions necessarily limit themselves to the case where the symmetric fusion category is Tannakian.

To produce a G-crossed braided category from a braided fusion category $\mathcal{C}$ containing a symmetric fusion category equivalent to $\mathrm{Rep}(G)$, one observes that $\mathrm{Rep}(G)$ contains the group algebra $\mathbb{C}[G]$ as algebra object. As $\mathrm{Rep}(G)$ is a braided monoidal subcategory of $\mathcal{C}$, this will also be an algebra object in $\mathcal{C}$, and it makes sense to talk about module objects for this algebra object. The category of such module objects internal to $\mathcal{C}$ is called the de-equivariantisation of $\mathcal{C}$. This category carries a G-action coming from the action of G on $\mathbb{C}[G]$. Furthermore, it is G-graded, with G-grading determined from the braiding behaviour of the module objects with the group algebra in $\mathcal{C}$. The action of the group will conjugate this grading. The tensor product over $\mathbb{C}[G]$ makes this category into a tensor category, and it carries a braiding up to the action of the group,.

In the particular case where the symmetric fusion category is the Müger centre of the ribbon fusion category $\mathcal{C}$, the de-equivariantisation is a modular tensor category known as the modularisation of $\mathcal{C}$.

Conversely, given a G-crossed braided fusion category, one can form a braided fusion category containing $\mathrm{Rep}(G)$ by taking the homotopy fixed points for the action of G. This procedure is known as equivariantisation. We remind the reader that taking homotopy fixed points on the level of objects means finding objects c that admit, for each $g \in G$, an isomorphism between c and the image under the action of g of c, subject to coherence conditions. For the subcategory spanned by the monoidal unit, the homotopy fixed points are exactly $\mathrm{Rep}(G)$. This category inherits a monoidal stucture, which is now honestly braided, the action of the group on the resulting category has been trivialised.

(De)-equivariantisation can be viewed as giving a 2-equivalence between appropriate 2-categories of braided fusion categories containing $\mathrm{Rep}(G)$ and of G-crossed braided fusion categories. This allows one to transport the degreewise tensor product of G-crossed braided categories to the 2-category of braided fusion categories containing $\mathrm{Rep}(G)$. The tensor product obtained in this way agrees with the reduced tensor product defined in this thesis.

In this thesis we give a factorisation of the above mutually inverse 2-functors into two pairs of mutually inverse 2-functors through an intermediate 2-category. The first 2-functor (and its inverse) in passing from braided fusion categories containing a symmetric fusion category to the intermediate 2-category does not require knowledge of Tannaka duality, and hence does not distinguish between the Tannakian and super-Tannakian case. Furthermore, the intermediate 2-category carries a natural symmetric monoidal structure, that we use to define the reduced tensor product for braided fusion categories. The second step necessarily involves Tannaka duality, but allows a uniform description for both cases. The super-Tannakian case gives rise to the notion of a super-G-crossed braided fusion category, which, to the author's knowledge, is new. We further show that the symmetric monoidal structure on the intermediate category is taken to the degreewise tensor product.

Categorical Perspective

A further motivation for the factorisation mentioned in the previous paragraph is the following. The first step in the factorisation is done purely in terms of monoidal structure and the braiding for the braided fusion category. In particular, we never need to find algebra objects and modules for these. This means that the expression we find for the reduced tensor product is directly in terms of the data of the monoidal structure and the braiding, making it more amenable to computations. This is in line with the philosophical argument that one should be able to construct such a product of categories in terms of just the categorical data.

When the symmetric fusion category is contained in the Müger centre of the ribbon fusion category and Tannakian, de-equivariantisation produces a ribbon fusion category, called the modularisation. Following the procedure outlined in this thesis gives a way of finding the de-equivariantisation without using the full force of Tannaka duality, we only make use of the fibre functor, not the finite group. Of course, when finding the equivariantisation of a category obtained in this way, one does need the finite group.

1.2 The Reduced Tensor Product

We will now outline our construction of the reduced tensor product of braided fusion categories containing a symmetric fusion category. Throughout, we fix two braided fusion categories $\mathcal{C}$ and $\mathcal{D}$ that both contain a symmetric fusion category $\mathcal{A}$.

1.2.1 The Setup

Our goal is to define a tensor product of $\mathcal{C}$ and $\mathcal{D}$, which satisfies the properties alluded to in the motivation. We will first collect these properties, and then discuss why some naive candidates for the reduced tensor product will not satisfy these properties, further motivating our construction.

Desiderata for the Reduced Tensor Product

From the first part of this Introduction, we see that we want to produce a tensor product $\overset{\mathcal{A}}{\underset{\text{red}}{\boxtimes}}$ of $\mathcal{C}$ and $\mathcal{D}$ that:

- outputs a braided fusion category containing $\mathcal{A}$,
- has the Drinfeld centre $\mathcal{Z}(\mathcal{A})$ as unit,
- can be computed in terms of the braiding and tensor structure of $\mathcal{C}$ and $\mathcal{D}$.

Existing candidates

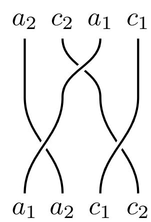
The first existing construction that one might consider using is the Deligne tensor product of linear categories, equipped with the componentwise tensor product and braiding. While this is certainly a braided fusion category and computable in terms of the braiding and tensor structure of $\mathcal{C}$ and $\mathcal{D}$, its unit is the category of vector spaces, rather than $\mathcal{Z}(\mathcal{A})$.

Another relevant construction is that of the balanced tensor product of module categories. As $\mathcal{A}$ is a tensor subcategory of $\mathcal{C}$ and $\mathcal{D}$, these categories can be viewed as $\mathcal{A}$-bimodule categories. For module categories, there exists the notion of balanced tensor product $\mathcal{C} \underset{\mathcal{A}}{\boxtimes} \mathcal{D}$, which is a categorification of the notion of tensor products of modules over an algebra. Indeed, objects of this balanced tensor product can be viewed as pairs $c \boxtimes d$ where c and d are objects of $\mathcal{C}$ and $\mathcal{D}$, respectively, which satisfy $(ca) \boxtimes d \cong c \boxtimes (ad)$ for $a \in \mathcal{A}$. There are several ways of computing the balanced tensor product, the approach relevant to this thesis is to enrich both $\mathcal{C}$ and $\mathcal{D}$ over $\mathcal{A}$. That is, we add to the hom-object between c and c' the vector space of morphisms $ac \to c'$ as a-summand. These hom-objects $\underleftarrow{\mathcal{C}}(c, c')$ have the defining property

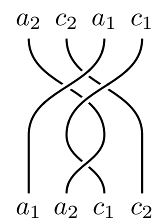
$$\mathcal{A}(a, \underleftarrow{\mathcal{C}}(c, c')) \cong \mathcal{C}(ac, c').$$

We will denote the $\mathcal{A}$-enriched categories obtained in this way by $\underleftarrow{\mathcal{C}}$ and $\underleftarrow{\mathcal{D}}$, respectively. For an object $a \in \mathcal{A}$, an a-point of a hom-object will be referred to as a degree a morphism. The balanced tensor product $\mathcal{C} \underset{\mathcal{A}}{\boxtimes} \mathcal{D}$ is obtained by taking as objects pairs of objects in $\mathcal{C}$ and $\mathcal{D}$ and the tensor product in $\mathcal{A}$ on the enriched hom-objects, and then applying the functor $\mathcal{A}(\mathbb{I}_{\mathcal{A}}, -)$ that picks out the monoidal unit summands to the hom-objects.

The category $\mathcal{C} \underset{\mathcal{A}}{\boxtimes} \mathcal{D}$ is a tensor category for the componentwise tensor product. It is no longer braided, unless $\mathcal{A}$ is contained in the Müger centres of $\mathcal{C}$ and $\mathcal{D}$. This is a consequence of the categories $\underleftarrow{\mathcal{C}}$ and $\underleftarrow{\mathcal{D}}$ not being braided themselves, the additional morphisms break the naturality of the braiding. This failure of the braiding to be natural is, for the tensor product of a degree a_1 morphism out of c_1 and a degree a_2 morphism out of c_2, exactly the difference between



and



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These diagrams differ from each other exactly by a pre-composition with the braiding monodromy $\beta_{c_1,a_2} \circ \beta_{a_2,c_1}$ between a_2 and c_1.

Recall that a braiding for a monoidal object in a symmetric monoidal 2-category is by definition a natural transformation between $- \otimes -$ and the switch functor for the symmetric monoidal 2-category followed by $- \otimes -$. The idea for fixing this failure of naturality of the braiding is now to encode the braiding monodromies into the switch functor.

1.2.2 Encoding the braiding monodromy

From our discussion of the balanced tensor product, we have learned that we need to keep track of the braiding behaviour of the objects of $\mathcal{C}$ and $\mathcal{D}$ with the objects of $\mathcal{A}$. We propose to do this by equipping the hom-objects in the $\mathcal{A}$-enrichment of $\mathcal{C}$ and $\mathcal{D}$ with a half-braiding for objects of $\mathcal{A}$, thus viewing them as objects in the Drinfeld centre $\mathcal{Z}(\mathcal{A})$ of $\mathcal{A}$. That is, we want to build $\mathcal{Z}(\mathcal{A})$-enriched categories out of $\mathcal{C}$ and $\mathcal{D}$.

If we pick the half-braidings so that they correspond to the braiding monodromies, using a switch functor that uses these half-braidings will indeed incorporate the braiding monodromies into the switch functor.

Enriching over the Drinfeld Centre

To capture the braiding monodromies in the half-braidings, we use a technical trick. Our category $\underleftarrow{\mathcal{C}}$ is enriched and tensored over $\mathcal{A}$, and as a consequence of this we have

$$a\underleftarrow{\mathcal{C}}(c, c') \cong \underleftarrow{\mathcal{C}}(c, ac').$$

The right hand object has an automorphism given by post-composition with the braiding monodromy of a and c'. We combine this with the symmetry in $\mathcal{A}$ to define the half-braiding $\beta_a \colon a\underleftarrow{\mathcal{C}}(c, c') \to \underleftarrow{\mathcal{C}}(c, c')a$ on $\underleftarrow{\mathcal{C}}(c, c')$. We will denote the object in $\mathcal{Z}(\mathcal{A})$ defined in this way by $\underleftarrow{\mathcal{C}}(c, c')$.

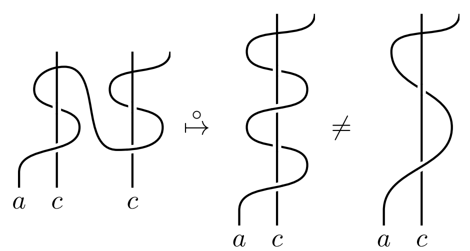
Having done this, the question is whether this assignment produces a $\mathcal{Z}(\mathcal{A})$-enriched monoidal category. In particular, we would like for the composition and monoidal structure of $\mathcal{C}$ to induce the composition and monoidal structure for $\underleftarrow{\mathcal{C}}$.

Composition

An illustrative minimal requirement for compatibility with composition is the desire for identities to compose. Recall that, in an enriched category, composition is a morphism out of the monoidal product in the enriching category. That is, if $\mathcal{K}$ is a category enriched over a monoidal category $(\mathcal{V}, \cdot)$, the composition in $\mathcal{K}$ is for objects $k, k', k'' \in \mathcal{K}$ a morphism:

$$\circ \colon \mathcal{K}(k', k'') \cdot \mathcal{K}(k, k') \to \mathcal{K}(k, k'').$$

The monoidal structure on the Drinfeld centre combines the half-braidings, as mentioned in Section 1.1.2. So the composite of endomorphisms on c would be a map out of an object equipped with the square of the braiding monodromies with objects $a \in \mathcal{A}$ as half-braidings into an object equipped with half-braiding coming from just the braiding monodromy. Schematically, for the identity on c, in string diagrams this is saying:



Here, we have used unresolved crossings to indicate the use of the symmetry in $\mathcal{A}$, this is an external operation that does not take place in $\mathcal{C}$. This use of the string diagram calculus is explained in Chapter 5. As these half-braidings will be different in general, this implies composition cannot factor through the $\otimes_c$ monoidal structure on $\mathcal{Z}(\mathcal{A})$.

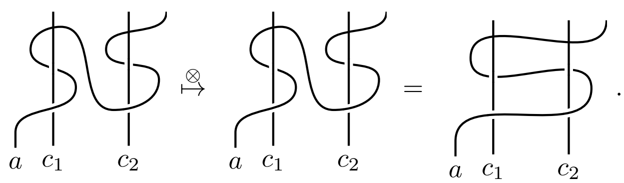
We will solve this problem by defining a second monoidal structure $\otimes_s$ on $\mathcal{Z}(\mathcal{A})$, that picks out the subobject of the tensor product of the underlying objects in $\mathcal{A}$ on which the half-braidings agree. We will then use this monoidal structure to factor the composition in our $\mathcal{Z}(\mathcal{A})$-enriched categories over.

Monoidal Structure

We also want our $\mathcal{Z}(\mathcal{A})$-enrichment to be compatible with the monoidal structure coming from the monoidal structure on $\mathcal{C}$. It turns out that the monoidal structure will factor through the $\otimes_c$-product on the hom-objects, i.e. define for objects $c_1, c_1', c_2, c_2' \in \mathcal{C}$ a map:

$$\otimes\colon \underleftarrow{\mathcal{C}}(c_1, c_1') \otimes_c \underleftarrow{\mathcal{C}}(c_2, c_2') \to \underleftarrow{\mathcal{C}}(c_1 \otimes c_2, c_1' \otimes c_2').$$

For identities on c_1 and c_2 this looks like:



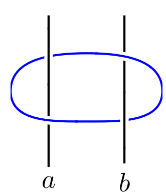
We see that the monoidal structure does factor over $\otimes_c$. This means we have to define a notion of a category that is enriched over $\mathcal{Z}(\mathcal{A})$ equipped with the second monoidal structure $\otimes_s$, but has a kind of monoidal structure that uses $\otimes_c$ on hom-objects. This will lead us to the notion of a $\mathcal{Z}(\mathcal{A})$-crossed braided category.

1.2.3 $\mathcal{Z}(\mathcal{A})$-Crossed Braided Categories

To capture the behaviour observed above, we want to find a notion of product $\underset{c}{\boxtimes}$ of $(\mathcal{Z}(\mathcal{A}), \otimes_s)$-enriched category that uses $\otimes_c$ on hom-objects, rather than $\otimes_s$ that we can define our monoidal structure out of. It turns out that what we need for this is for $\otimes_s$ and $\otimes_c$ to be laxly compatible.

The Symmetric Tensor Product on $\mathcal{Z}(\mathcal{A})$

The operation $\otimes_s$ on $\mathcal{Z}(\mathcal{A})$ is defined as follows. Given two objects (a, β^a) and (b, β^b) in $\mathcal{Z}(\mathcal{A})$, we want to first pick out the subobject of $a \otimes_{\mathcal{A}} b$ on which the half-braidings β^a and β^b "agree". To do this, we use the idempotent:

$\Pi_{a,b} =$  .

The ring represents a sum over the simple objects of $\mathcal{A}$, this idempotent can be read as checking, for each of these simples, if the half-braidings will cancel each other out. Now we take the subobject associated to this idempotent, and we equip it with the half-braiding coming from either (a, β^a) or (b, β^b), which on this subobject agree.

This tensor product is symmetric, with symmetry induced by the symmetry in $\mathcal{A}$.

$\mathcal{Z}(\mathcal{A})$ as a 2-fold Tensor Category

By the well-known Eckmann-Hilton argument, any two monoid structures on a set that are homomorphisms with respect to each other are equal and commutative. This argument generalises to monoidal structures on categories, if two monoidal structures are strongly monoidal with respect to each other, they must agree. However, if we allow the comparison natural transformations to be non-invertible, i.e. we ask for the monoidal structures to be only laxly compatible, this argument fails. This means that we have a notion of 2-fold monoidal category as being a category equipped with two laxly compatible monoidal structures.

From the way we construct the symmetric tensor product $\otimes_s$, we expect there to be a relationship between $\otimes_s$ and $\otimes_c$. Specifically, it turns out that there is a morphism

$$(a \otimes_s b) \otimes_c (c \otimes_s d) \to (a \otimes_c c) \otimes_s (b \otimes_c d),$$

that exhibits $(a \otimes_s b) \otimes_c (c \otimes_s d)$ as a subobject of $(a \otimes_c c) \otimes_s (b \otimes_c d)$ in $\mathcal{Z}(\mathcal{A})$ for objects $a, b, c, d \in \mathcal{Z}(\mathcal{A})$.

These morphisms combine to a lax compatibility natural transformation, and the associated projections into an oplax compatibility. This makes the Drinfeld centre of $\mathcal{A}$ into a 2-fold tensor category. Furthermore, the symmetry for $\otimes_s$ and the braiding for $\otimes_c$ are compatible with this lax structure.

$\mathcal{Z}(\mathcal{A})$-Crossed Tensor Categories

Due to the lax compatibility between $\otimes_s$ and $\otimes_c$, we can define a tensor product $\mathcal{K} \underset{c}{\boxtimes} \mathcal{L}$ of $(\mathcal{Z}(\mathcal{A}), \otimes_s)$-enriched categories $\mathcal{K}$ and $\mathcal{L}$ by taking the objects to be pairs $k \boxtimes l$ of objects in $\mathcal{K}$ and $\mathcal{L}$, while taking $\otimes_c$ on hom-objects. This is the tensor product of categories our monoidal structure will factor through.

A $\mathcal{Z}(\mathcal{A})$-crossed tensor category is then a $(\mathcal{Z}(\mathcal{A}), \otimes_s)$-enriched category with monoidal structure $\otimes \colon \mathcal{K} \underset{c}{\boxtimes} \mathcal{K}$. The categories $\underset{\leftarrow}{\mathcal{C}}$ and $\underset{\leftarrow}{\mathcal{D}}$ obtained by enriching over $\mathcal{Z}(\mathcal{A})$ are $\mathcal{Z}(\mathcal{A})$-crossed tensor categories.

$\mathcal{Z}(\mathcal{A})$-Crossed Braided Categories

The product $\underset{c}{\boxtimes}$ comes with a switch functor induced by the braiding for $\otimes_c$. Using this switch functor we can now define a $\mathcal{Z}(\mathcal{A})$-crossed braided category to be a $\mathcal{Z}(\mathcal{A})$-crossed tensor category equipped with a monoidal natural isomorphism between the composite of this switch functor with the monoidal structure and the monoidal structure. We further require this braiding to satisfy the usual hexagon equations.

The categories $\underset{\leftarrow}{\mathcal{C}}$ and $\underset{\leftarrow}{\mathcal{D}}$ are $\mathcal{Z}(\mathcal{A})$-crossed braided categories, as having incorporated the braiding monodromies into the half-braiding ensures the original braidings will be natural.

Tensor Product of $\mathcal{Z}(\mathcal{A})$-Crossed Braided Categories

For categories enriched over a symmetric monoidal category, there is a symmetric tensor product of such categories. This is obtained by taking objects to be pairs and using the monoidal structure on hom-objects. For the case of $(\mathcal{Z}(\mathcal{A}), \otimes_s)$-enriched categories, this means using the symmetric monoidal structure $\otimes_s$, we will denote this product by $\underset{s}{\boxtimes}$.

Using the oplax compatibility between $\otimes_s$ and $\otimes_c$ we can show that, for $\mathcal{Z}(\mathcal{A})$-crossed braided categories $\mathcal{K}$ and $\mathcal{L}$, the product $\mathcal{K} \underset{s}{\boxtimes} \mathcal{L}$ is again $\mathcal{Z}(\mathcal{A})$-crossed braided. Furthermore, the unit for $\underset{s}{\boxtimes}$ is the category $\underleftarrow{\mathcal{Z}(\mathcal{A})}$ obtained by applying the enriching procedure described above to $\mathcal{Z}(\mathcal{A})$ itself.

We recognise $\underset{s}{\boxtimes}$ as coming close to the desiderata outlined for the reduced tensor product. We have found a product that outputs a braided category, with a version of $\mathcal{Z}(\mathcal{A})$ as the unit, and which can be computed in terms of the braiding and monoidal structure. All that is left now is to produce from $\underleftarrow{\mathcal{C}} \underset{s}{\boxtimes} \underleftarrow{\mathcal{D}}$ a braided fusion category.

De-enriching

The procedure outlined above to obtain from a braided fusion category containing $\mathcal{A}$ a $\mathcal{Z}(\mathcal{A})$-crossed braided category admits an inverse. This inverse will be called de-enriching, and is performed in two steps. First, one forgets the half-braidings on the hom-objects, returning to a $\mathcal{A}$-enriched category. Then, one applies the functor $\mathcal{A}(\mathbb{I}_{\mathcal{A}}, -)$ to the hom-objects to obtain a linear category. The 2-functor this construction defines will be denoted by **DeEnrich**.

We will show that de-enriching a $\mathcal{Z}(\mathcal{A})$-crossed braided fusion category gives a braided fusion category, and that, in fact, de-enriching $\underleftarrow{\mathcal{C}}$ gives back $\mathcal{C}$.

1.2.4 The Reduced Tensor Product

We have now described all the ingredients needed to define the reduced tensor product of braided fusion categories containing $\mathcal{A}$.

Definition of the Reduced Tensor Product

Given braided fusion categories $\mathcal{C}$ and $\mathcal{D}$ containing a symmetric fusion category $\mathcal{A}$, we can now define the reduced tensor product of $\mathcal{C}$ and $\mathcal{D}$ by:

$$\mathcal{C} \overset{\mathcal{A}}{\underset{\text{red}}{\boxtimes}} \mathcal{D} := \mathbf{DeEnrich}(\underleftarrow{\mathcal{C}} \underset{s}{\boxtimes} \underleftarrow{\mathcal{D}}).$$

The unit will be given by $\mathcal{Z}(\mathcal{A})$.

1.3 Results

We now summarise the results needed to follow the above route, and the additional results we prove along the way.

1.3.1 Symmetric Tensor Product on the Drinfeld Centre

In Chapter 2, we discuss how to define the symmetric tensor product $\otimes_s$ on the Drinfeld centre of a symmetric fusion category $\mathcal{A}$. We then examine what this symmetric tensor product looks like in the Tannakian and super-Tannakian cases.

The Symmetric Tensor Product

We prove that the symmetric tensor product is a symmetric tensor product on $\mathcal{Z}(\mathcal{A})$.

Theorem A. *The Drinfeld centre equipped with the symmetric tensor product* $(\mathcal{Z}(\mathcal{A}), \otimes_s, \mathbb{I}_s, s)$ *is a symmetric tensor category.*

This appears as Theorem 2.22 in the main text. The definition of $\otimes_s$ on objects is given in Definition 2.11, on morphisms it is defined in Definition 2.22. The unit $\mathbb{I}_s$ is given in Definition 2.16. The symmetry s is induced by the symmetry in $\mathcal{A}$ and is described in Lemma 2.13.

The Tannakian Case

For the case where $\mathcal{A} \cong \mathrm{Rep}(G)$, we show that the symmetric tensor product is sent to the fibrewise tensor product of G-equivariant vector bundles over G under the equivalence between the Drinfeld centre of $\mathrm{Rep}(G)$ and the category $\mathbf{Vect}_G[G]$.

Theorem B. *Let G be a finite group. Then the equivalence*

$$\mathcal{Z}(\mathrm{Rep}(G)) \cong \mathbf{Vect}_G[G]$$

takes the symmetric tensor product $\otimes_s$ to the fibrewise tensor product of G-equivariant vector bundles.

We prove this as Theorem 2.28.

The Super-Tannakian Case

In the super-Tannakian case where $\mathcal{A}$ is equivalent to the category $\mathrm{Rep}(G, \omega)$ of a finite super-group (G, ω), we define, for each central order two element $\omega \in G$ a super version of the fibrewise tensor product:

Definition C (Definition 2.33)**.** Let G be a finite group, and pick a central order two element $\omega \in G$. This choice induces a $\mathbb{Z}_2$-grading obtained by viewing ω as grading involution on G-equivariant vector bundles over G. The *fibrewise super-tensor product* for this choice of ω of two G-equivariant vector bundles V and W over G that are homogeneous for the $\mathbb{Z}_2$-grading has fibres:

$$(V \otimes_f^\omega W)_g = V_{\omega^{|W|}g} W_{\omega^{|V|}g},$$

where $|V|$ denotes the grading of V, and the product is the tensor product of vector spaces.

The author believes this Definition has not previously appeared in the literature.

This super-version of the fibrewise tensor product is exactly the image of the symmetric tensor product under the equivalence $\mathcal{Z}(\mathrm{Rep}(G,\omega)) \cong \mathbf{Vect}_G[G]$:

Theorem D (Theorem 2.35)**.** *Let (G,ω) be a finite super-group. Then the equivalence*

$$\mathcal{Z}(\mathrm{Rep}(G,\omega)) \cong \mathbf{Vect}_G[G]$$

takes the symmetric tensor product $\otimes_s$ to the fibrewise super-tensor product $\otimes_f^\omega$.

1.3.2 The Drinfeld Centre as a 2-Fold Tensor Category

In Chapter 3, we show that the symmetric tensor product $\otimes_s$ from Chapter 2 together with the usual tensor product $\otimes_c$ gives $\mathcal{Z}(\mathcal{A})$ the structure of a 2-fold tensor category.

The Notion of Lax Compatibility

The lax compatibility between $\otimes_s$ and the usual tensor tensor product $\otimes_c$ on the Drinfeld centre of the symmetric fusion category $\mathcal{A}$ has some additional desirable structure. For example, post-composing the oplax comparison with the lax comparison gives the identity, as they come from inclusions and projections for the same subobjects. To capture this, we make the following definition. Recall that a 2-fold tensor category is a linear category with two laxly compatible tensor structures.

Definition E (Definitions 3.1, 3.2, 3.3 and 3.4)**.** A 2-fold tensor category is called *vertically symmetric braided strongly inclusive bilax* if the tensor structures are also oplax compatible, the oplax comparison composed with the lax comparison is the identity, the product of the monoidal for the first tensor structure with itself for second tensor structure is isomorphic to this unit, both structures are compatibly braided, and the braiding for the second product is symmetric.

The Drinfeld Centre as 2-Fold Tensor Category

The main result of Chapter 3 is:

Theorem F. *The category $(\mathcal{Z}(\mathcal{A}), \otimes_c, \otimes_s)$ is a vertically symmetric braided strongly inclusive bilax 2-fold tensor category.*

This appears as Theorem 3.5.

1.3.3 $\mathcal{Z}(\mathcal{A})$-Crossed Braided Tensor Categories

Chapter 4 is devoted to developing the theory of $\mathcal{Z}(\mathcal{A})$-crossed braided tensor categories.

Definition of $\mathcal{Z}(\mathcal{A})$-Crossed Braided Categories

The definition of $\mathcal{Z}(\mathcal{A})$-crossed braided categories appears in the main text as Definitions 4.16 and 4.20. These combine to:

Definition G. A $(\mathcal{Z}(\mathcal{A}), \otimes_s)$-enriched category $\mathcal{K}$ is called a $\mathcal{Z}(\mathcal{A})$-*crossed braided tensor category* if it is equipped with a monoidal structure

$$\otimes : \mathcal{K} \underset{c}{\boxtimes} \mathcal{K} \to \mathcal{K},$$

that is braided with respect to the switch functor for $\underset{c}{\boxtimes}$ that is induced by the half-braiding in $\mathcal{Z}(\mathcal{A})$. The unit is defined to be a functor $\mathcal{A}_{\mathcal{Z}} \to \mathcal{K}$ satisfying the usual conditions.

Here $\underset{c}{\boxtimes}$ is the tensor product of $(\mathcal{Z}(\mathcal{A}), \otimes_s)$-enriched categories that has objects pairs of objects and takes the tensor product $\otimes_c$ on hom-objects. The category $\mathcal{A}_{\mathcal{Z}}$ is $\mathcal{A}$ enriched over $\mathcal{A} \subset \mathcal{Z}(\mathcal{A})_s$, this is the unit for $\underset{c}{\boxtimes}$. These categories also admit a symmetric monoidal product of categories $\underset{s}{\boxtimes}$ obtained by using $\otimes_s$ on hom-objects.

Equivalence with (super) G-Crossed Braided Categories

Using Tannaka duality, we can study the relation between $\mathcal{Z}(\mathcal{A})$-crossed braided categories and the existing notion of G-crossed braided tensor categories. In the super-Tannakian case, this leads to the notion of super-(G, ω)-crossed braided tensor category (Definition 4.31), which to the author's knowledge is new.

After setting up 2-categories $\mathcal{Z}(\mathcal{A})$-**XBF** of $\mathcal{Z}(\mathcal{A})$-crossed braided tensor categories and G-**XBF** (or (G, ω)-**XBF**) of (super-)G-crossed braided tensor categories, we prove:

Theorem H (4.27)**.** *There is a symmetric monoidal equivalence of 2-categories*

$$\overline{\overline{(-)}} \colon \mathcal{Z}(\mathcal{A})\text{-}\mathbf{XBF} \longleftrightarrow G\text{-}\mathbf{XBF} : \mathbf{Fix},$$

*where we replace G-**XBF** by (G, ω)-**XBF** in the super-Tannakian case.*

The 2-functor $\overline{\overline{(-)}}$ is induced from the composite of the forgetful functor $\mathcal{Z}(\mathcal{A}) \to \mathcal{A}$ with the (super)-fibre functor $\mathcal{A} \to \mathbf{sVect}$. The 2-functor **Fix** takes homotopy fixed points for the G-action on the G-crossed braided categories.

1.3.4 From Braided Fusion Categories to $\mathcal{Z}(\mathcal{A})$-Crossed Braided Categories

Having set up the theory of $\mathcal{Z}(\mathcal{A})$-crossed braided categories, in Chapter 5 we show how to obtain these categories from braided fusion categories containing $\mathcal{A}$. We will show that this construction has an inverse.

Enriching over the Drinfeld Centre

The first part Chapter 5 is devoted to showing how to obtain $\mathcal{Z}(\mathcal{A})$-crossed braided categories from braided fusion categories containing $\mathcal{A}$, we will call this procedure $\underleftarrow{(-)}$. We prove:

Theorem I (Theorem 5.8)**.** *Let $\mathcal{C}$ be a braided fusion category containing a symmetric fusion category $\mathcal{A}$, then the category $\underleftarrow{\mathcal{C}}$ is a $\mathcal{Z}(\mathcal{A})$-crossed braided category.*

We set up a 2-category of braided fusion categories containing $\mathcal{A}$, denoted $\mathbf{BFC}/\mathcal{A}$ and show that $\underleftarrow{(-)}$ defines a 2-functor from this 2-category to the 2-category to $\mathcal{Z}(\mathcal{A})$-$\mathbf{XBF}$.

Equivalence between $\mathbf{BFC}/\mathcal{A}$ and $\mathcal{Z}(\mathcal{A})$-$\mathbf{XBF}$

The 2-functor $\underleftarrow{(-)}$ admits an inverse, given by **DeEnrich**. This 2-functor is induced from the composite of the forgetful functor $\mathcal{Z}(\mathcal{A}) \to \mathcal{A}$ with $\mathcal{A}(\mathbb{I}_s, -)$.

Theorem J. *There is an equivalence of 2-categories*

$$\underleftarrow{(-)} \colon \mathbf{BFC}/\mathcal{A} \longleftrightarrow \mathcal{Z}(\mathcal{A})\text{-}\mathbf{XBF} : \mathbf{DeEnrich}.$$

This is Theorem 5.41 in this thesis.

1.3.5 Relation to De-Equivariantisation

The results above are intimately related to (de-)equivariantisation, as we will explore in Chapter 6. Their relationship is expressed in the following diagram:

$$\begin{tikzcd}
 & \mathcal{Z}(\mathcal{A})\text{-}\mathbf{XBF} \arrow[dl, shift left, "\mathbf{DeEnrich}"] \arrow[dr, shift left, "\overline{\overline{(-)}}"] & \\
\mathbf{BFC}/\mathcal{A} \arrow[ur, shift left, "\underleftarrow{(-)}"] \arrow[rr, shift left, "\mathbf{De{-}Eq}"] & & G\text{-}\mathbf{XBF}. \arrow[ul, shift left, "\mathbf{Fix}"] \arrow[ll, shift left, "\mathbf{Eq}"]
\end{tikzcd}$$

Here $\mathbf{De} - \mathbf{Eq}$ is the 2-functor defined by de-equivariantisation, which takes a braided fusion category $\mathcal{C}$ containing $\mathcal{A} = \mathrm{Rep}(G)$ to the category of modules over $\mathbb{C}[G]$ internal to $\mathcal{C}$. Its inverse, $\mathbf{Eq}$, takes homotopy fixed points. We show that this diagram commutes in two steps.

Equivariantisation

We prove the following as Theorem 6.4.

Theorem K. *The composite*

$$G\text{-}\mathbf{XBF} \xrightarrow{\mathbf{Fix}} \mathcal{Z}(\mathcal{A})\text{-}\mathbf{XBF} \xrightarrow{\mathbf{DeEnrich}} \mathbf{BFC}/\mathcal{A}$$

is equal to $\mathbf{Eq}$.

De-Equivariantisation

As the diagram above commutes for the left-pointing arrows, and all these are inverses to the right pointing arrows, we obtain:

Theorem L (Corollary 6.5)**.** *The composite*

$$\mathbf{BFC}/\mathcal{A} \xrightarrow{\underleftarrow{\overline{(-)}}} \mathcal{Z}(\mathcal{A})\text{-}\mathbf{XBF} \xrightarrow{\overline{\overline{(-)}}} G\text{-}\mathbf{XBF}$$

is equivalent to $\mathbf{De} - \mathbf{Eq}$.

1.3.6 The Reduced Tensor Product

The above constructions allow us to define the reduced tensor product in Chapter 6.

Definition of the Reduced Tensor Product

Using the (de)-enriching construction from Chapter 5, we can define:

Definition M. Let $\mathcal{C}$ and $\mathcal{D}$ be braided fusion categories containing $\mathcal{A}$. Then the *reduced tensor product* of $\mathcal{C}$ and $\mathcal{D}$ is given by:

$$\mathcal{C} \overset{\mathcal{A}}{\underset{\mathrm{red}}{\boxtimes}} \mathcal{D} := \mathbf{DeEnrich}(\underleftarrow{\underline{\mathcal{C}}} \underset{s}{\boxtimes} \underleftarrow{\underline{\mathcal{D}}}).$$

Properties of the Reduced Tensor Product

With this definition of the reduced tensor product we can establish:

Theorem N. *The 2-category* $\mathbf{BFC}/\mathcal{A}$ *is symmetric monoidal for* $\overset{\mathcal{A}}{\underset{\mathrm{red}}{\boxtimes}}$*, with unit given by* $\mathcal{Z}(\mathcal{A})$*.*

This combines Theorem 5.41 with Proposition 6.7.

Minimal Modular Extensions

Denote the set of minimal modular extensions of $\mathcal{C}$ with $\mathcal{Z}_2(\mathcal{C}) = \mathcal{A}$ by $\mathrm{MME}_{\mathcal{A}}(\mathcal{C})$.

Theorem O (Corollary 6.18)**.** *The reduced tensor product gives a pairing:*

$$- \overset{\mathcal{A}}{\underset{\mathrm{red}}{\boxtimes}} - \colon \mathrm{MME}_{\mathcal{A}}(\mathcal{C}) \times \mathrm{MME}_{\mathcal{A}}(\mathcal{D}) \to \mathrm{MME}_{\mathcal{A}}(\mathcal{C} \underset{\mathcal{A}}{\boxtimes} \mathcal{D}).$$

1.4 Related work

We will now discuss existing work related to the contents of this thesis. This splits roughly into two parts. The first is the fusion category literature. Fusion categories have been extensively studied since their conception. The second part is literature more closely related to the applications of fusion categories in physics.

1.4.1 Fusion Categories

The basic reference for the theory of fusion categories is [ENO05]. This paper develops the basic theory of fusion categories. For a treatment in terms of string diagrams, which are used extensively in this thesis, see [Bar16].

Tannaka Duality

Deligne's original proof of the Tannaka duality used in this paper is spread out over [Del90, Del02]. A review is in [Ost04].

Modularisation

The factorisation of de-equivariantisation presented in this thesis is inspired by and closely related to [Müg00]. In this article the modularisation (called modular closure there) is constructed by first defining new hom-objects that, from our point of view, are the image under the fibre functor for the symmetric fusion category. Upon performing idempotent completion one then obtains the modularisation. For a treatment of the theory of modular tensor categories see [Müg03b].

(De)-Equivariantisation

(De-)Equivariantisation is discussed in detail in [DGNO10]. This paper is also a standard reference for facts about braided fusion categories. The relation between de-equivariantisation and G-crossed braided categories is summarised in [Müg10], which also contains detailed references to the literature on this subject.

The Reduced Tensor Product

The definition of the reduced tensor product given here owes its name, and its main inspiration to an unpublished note by Drinfeld [Drib]. This note defines the reduced tensor product in the special case where $\mathcal{A} = \mathbf{sVect}$. This note is closely related to the discussion of Ising categories in [DGNO10, Appendix B].

1.4.2 Topological Quantum Field Theory

Topological quantum field theories have been widely studied since they were first defined by Atiyah [Ati88], partly based on Segal's treatment of conformal field theory [Seg88] and inspired by Witten's influential insight into the relation between Chern-Simons theory and the Jones polynomial [Wit89]. The work in this thesis is related to three-dimensional once-extended theories.

Once-extended Three-Dimensional TQFT

Since Witten's observations concerned three-dimensional field theories, three-dimensional topological quantum field theories are particularly well studied. The field roughly splits into the study of fully extended or state-sum theories, usually collectively called Turaev-Viro, and once-extended theories. The state-sum construction was first given in [TV92], for a treatment of these invariants from the perspective of fully extended field theories see [DSSP].

Once-extended three-dimensional topological quantum field theories were first described using a formalism known as modular functors. They were originally studied by Reshetikhin and Turaev [RT91] as a way of making Witten's observations about the Jones polynomial precise. For an extensive treatment of the techniques involved, see [BK01].

A modern perspective on this, from the point of view of symmetric monoidal pseudo-functors out of the bordism bicategory in three-dimensions, is given in [BDSPV14, BDSPV15]. These papers are part of a series that gives a complete proof of the one-to-one correspondence between modular tensor categories and (signature-extended) oriented once-extended three-dimensional field theories.

Field theories on bordisms equipped with principal G-bundles are studied from the modular functor perspective in [Tur10]. In this book Turaev also proves one can obtain a G-equivariant modular functor from a G-crossed modular tensor category (a particular type of G-crossed braided category).

Dijkgraaf-Witten Theory

A motivating example for many of the constructions done here is Dijkgraaf-Witten theory. This theory originates from Physics [DW90]. It is extensively studied in the Mathematics literature. A treatment close to the original formulation is in [Wil08]. For a functorial field theory perspective, see [FQ93].

Minimal Modular Extensions

Minimal modular extensions were first defined by Müger in [Müg03b], where it was conjectured that minimal modular extensions always exist. An example of a braided fusion category that does not admit a minimal modular extension was found by Drinfeld [Dria].

The connection between minimal modular extensions and short range entangled phases was first discussed in [LKW17a, LKW17b], where it is also shown that minimal modular extensions form a torsor for the minimal modular extensions of the Müger centre. The techniques used to find this torsor structure are based on finding so-called Lagrangian algebra objects. This thesis is in part motivated by a desire to understand this torsor structure directly in terms of the braiding and the modular structure.

An explicit construction of the categories obtained by using the torsor structure in the special case where the symmetric fusion category is **sVect** is given in [BGH+17].

Orbifolding in Field Theory

Another important idea behind this thesis is that of orbifolding, passing from a field theory with a G-action to a quotient, for us this corresponds to taking homotopy fixed points. In the context of fusion categories, this is discussed in [Kir02]. For a treatment of orbifolding in the context of topological quantum field theory, see [SW17].

1.5 Outlook

We will now discuss possibilities for future work based on this thesis.

1.5.1 The Reduced Tensor Product

There are some open questions regarding the reduced tensor product defined here.

Invertible Objects

In Proposition 6.13 we discuss what the invertible objects for $\overset{\mathcal{A}}{\underset{\text{red}}{\boxtimes}}$ are in the Tannakian case. Similar arguments should give a classification of the invertible objects in the super-Tannakian case. The arguments given require the use of Tannaka duality, the question is whether there is a description of the group of invertible objects that is independent of this. In particular, an original motivation for the project that gave rise to this thesis was to explain the coincidence of Kitaev's 16-fold way [Kit06] and Drinfeld's observation [Drib] that the reduced tensor product for the case $\mathcal{A} = \mathbf{sVect}$ has a cyclic group of order 16 as its invertible objects.

Minimal Modular Extensions

We have shown that the reduced tensor product gives a pairing, see Theorem O. We would like to use this to reproduce the result from [LKW17a] that the minimal modular extensions of a category $\mathcal{C}$ are, if non-empty, a torsor for the minimal modular extensions of its Müger centre.

The pairing of minimal modular extension would also be interesting to study on its own, as it allows us to find new minimal modular extensions from existing ones.

Modular Data

Ribbon fusion categories are frequently studied through their modular data, which encodes the fusion rules, braiding monodromies and twists. As the reduced tensor product is defined directly in terms of these structures, we would like to have a description of the reduced tensor in terms of the modular data. This is particularly appealing from a computational point of view, the modular data is used to find classification results for ribbon fusion categories of a given rank.

1.5.2 $\mathcal{Z}(\mathcal{A})$-Crossed Braided Categories

The notion of $\mathcal{Z}(\mathcal{A})$-crossed braided category presented in this thesis is new, and there are some questions related to studying what kind of objects these are.

Relation to Bundles of Categories

The notion of G-crossed braided category can be seen as a categorification of $\mathbf{Vect}_G[G]$, i.e. as bundles of linear categories over the action groupoid for the conjugation action G. We would like to know if there is a similar description of $\mathcal{Z}(\mathcal{A})$-crossed braided categories.

Modularity

Both 2-categories $\mathbf{BFC}/\mathcal{A}$ and G-$\mathbf{XBF}$ admit a definition of what it means for an object to be modular. There should be a corresponding definition of modularity for $\mathcal{Z}(\mathcal{A})$-$\mathbf{XBF}$. By [BDSPV15], the objects $\mathbf{BFC}/\mathcal{A}$ that are modular tensor categories correspond to $\mathbf{LinCat}$-valued once-extended three-dimensional (signature extended) oriented topological quantum field theories. $\mathcal{Z}(\mathcal{A})$-crossed modular categories should allow for a similar description, likely in terms of modular objects in the 2-category of $\mathcal{A}$-enriched and tensored categories.

1.5.3 G-Equivariant Field Theories and Orbifolding

Though this thesis is partly inspired by topological quantum field theory considerations, these considerations are not part of this work. This leaves room for future work.

From G-Equivariant to Oriented

One element central to our motivation, the passage from G-equivariant theories to oriented theories and its relation to taking homotopy fixed points for G-crossed braided categories, remains unexplored. It would be interesting to explore this relationship further. Steps in this direction have already been taken in [SW17].

The Reduced Tensor Product as Product of Field Theories

The reduced tensor product should come about as a product of field theories. To show this is indeed the case, one would need a classification of G-equivariant once-extended three-dimensional field theories.

Chapter 2

The Symmetric Tensor Product on the Drinfeld Centre of a Symmetric Fusion Category

2.1 Introduction

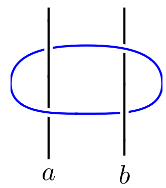
Let $(\mathcal{A}, \otimes)$ be a symmetric ribbon fusion category over $\mathbb{C}$. It is well-known [Müg03a] that its Drinfeld center $\mathcal{Z}(\mathcal{A})$ (Definition A.36) is a modular tensor category. By Tannaka duality [Del90] (see Theorem A.35), there is a finite group G (or super-group (G, ω)) such that $\mathcal{A} = \mathrm{Rep}(G)$ (or $\mathrm{Rep}(G, \omega)$). In the non-super case, with this identification, we have another description of the Drinfeld centre as the category $\mathbf{Vect}_G[G]$ of G-equivariant vector bundles on G (Definition A.37), equipped with the convolution tensor product. This category carries an additional tensor structure given by fibrewise tensor product, and this tensor structure is symmetric.

Our goal is to define a symmetric tensor product

$$\otimes_s \colon \mathcal{Z}(\mathcal{A}) \boxtimes \mathcal{Z}(\mathcal{A}) \to \mathcal{Z}(\mathcal{A}),$$

that is a purely categorical version of the fibrewise tensor product. We avoid using Tannaka duality in defining $\otimes_s$. In particular, this categorical description will treat the super and non-super Tannakian cases on equal footing. In the super Tannakian case, this will lead us to define a generalisation of the fibrewise tensor product to equivariant vector bundles over a super-group. Additionally, we will show in Chapter 3 that the symmetric tensor product $\otimes_s$ together with the usual tensor product $\otimes_c$ make the Drinfeld centre into a lax 2-fold tensor category.

To define $a \otimes_s b$ we will employ the idempotent on $a \otimes_c b$ (defined in Section 2.3.1) given by

$\Pi_{a,b} =$ 

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to pick out the subobject of $a \otimes b \in \mathcal{Z}(\mathcal{A})$ on which the half-braidings of a and b agree. We then equip this subobject with the half-braiding coming from (equivalently) a or b.

The outline of this Chapter is as follows. In Section 2.2 we introduce some notation and useful lemmas about subobjects in idempotent complete categories, and about string diagrams. Then, in Section 2.3, we will define the symmetric tensor product on $\mathcal{Z}(\mathcal{A})$. We will do this in two parts. First we will define the operation $\otimes_s$ on objects, and establish the associators, unit object and unitors, and symmetry objectwise. Secondly, we define $\otimes_s$ on morphisms and show that this definition makes $(\mathcal{Z}(\mathcal{A}), \otimes_s)$ into a symmetric monoidal category. In the final Section 2.4, we verify that, given a fibre functor on $\mathcal{A}$, $\otimes_s$ agrees with the fibrewise tensor product on $\mathbf{Vect}_G[G]$ in the Tannakian case. In the super-Tannakian case, where $\mathcal{A} = \mathrm{Rep}(G, \omega)$, we first define a new tensor product $\mathbf{Vect}_G[G]$ that depends on the choice of central element ω. We then show that the symmetric tensor product on $\mathcal{Z}(\mathcal{A})$ is taken to this tensor product on $\mathbf{Vect}_G$ under the equivalence $\mathcal{Z}(\mathcal{A}) \cong \mathbf{Vect}_G[G]$.

2.2 Preliminaries

2.2.1 Notation

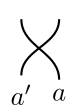
Throughout, we will suppress the associators of $\mathcal{A}$ (and hence of $\mathcal{Z}(\mathcal{A})$). When there is no risk of confusion, we will suppress the symbol $\otimes$. We will make use of the string diagram calculus for ribbon categories, reading the diagrams from bottom to top.

When drawing string diagrams in $\mathcal{Z}(\mathcal{A})$ we will take the convention that crossings correspond to braiding according to the half-braiding of the over-crossing object. That is, if $(a, \beta) \in \mathcal{Z}(\mathcal{A})$, with $\beta\colon - \otimes a \Rightarrow a \otimes -$, and $c \in \mathcal{Z}(\mathcal{A})$, we will denote:

$\beta_c =$ 

.

Unresolved crossings will denote the use of the symmetry s in $\mathcal{A}$. So for $(a, \beta), (a', \beta') \in \mathcal{Z}(\mathcal{A})$,

$s_{a',a} =:$ 

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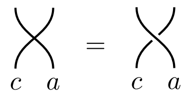
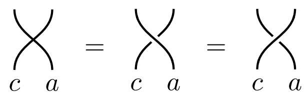
We will sometimes choose to resolve crossings between objects in $\mathcal{A} \subset \mathcal{Z}(\mathcal{A})$ and objects in $\mathcal{Z}(\mathcal{A})$, in order to make manipulations of the string diagrams easier to follow. Given $(a, s_{-,a}) \in \mathcal{A} \subset \mathcal{Z}(\mathcal{A})$ and $c \in \mathcal{Z}(\mathcal{A})$,

$s_{c,a} =:$


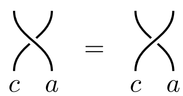
In the case where also $c = (a', s_{-,a'}) \in \mathcal{A} \subset \mathcal{Z}(\mathcal{A})$, we have:

$s_{a',a} =:$


(2.1)

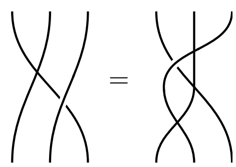
because in this case both half-braidings are given by the symmetry in $\mathcal{A}$. The following notion will be used throughout

Definition 2.1. Let $a, c \in \mathcal{C}$ be objects of a braided monoidal category. If

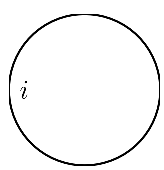


then a and c are said to be *transparent* to each other.

Because of the naturality and monoidality of the symmetry, the resolved and unresolved crossings satisfy:


(2.2)

In the rest of this thesis, we will denote a set of representatives of the isomorphism classes of simple objects of $\mathcal{A}$ by $\mathcal{O}(\mathcal{A})$. The quantum dimension of $i \in \mathcal{O}(\mathcal{A})$ will be denoted by

$d_i =$


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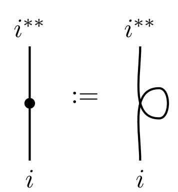
where the pivotal structure $i \cong i^{**}$ on the right hand side of the loop has been suppressed. We will also make use of the following notation:



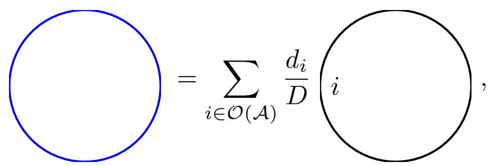
(2.3)

To make $\mathcal{A}$ into a ribbon category, we define composing this morphism with the pivotal structure to be the twist θ_i on i. From this we read off that, because $\mathcal{A}$ is symmetric, the twist will be $\pm\mathrm{id}$ on simple objects. The global dimension of $\mathcal{A}$ will be denoted by

$$D := \sum_{i \in \mathcal{O}(\mathcal{A})} d_i^2.$$

This global dimension will always be non-zero, as we are working with fusion categories over the complex numbers [ENO05, Theorem 2.3].

We will use the additional notation



(2.4)

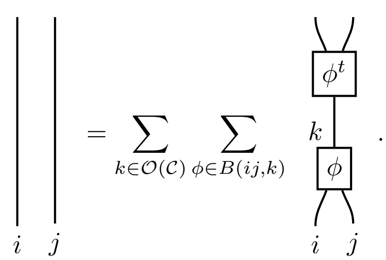
whenever we encounter an unlabelled loop in a string diagram.

2.2.2 Direct sum decompositions

In our proofs we will make frequent use of the following lemmas and notation. We will introduce them in the setting of a ribbon fusion category $\mathcal{C}$.

Notation 2.2. Given $i, j, k \in \mathcal{C}$, we will denote by $B(ij, k)$ a basis for the vector space $\mathcal{C}(ij, k)$.

Since $\mathcal{C}$ is in particular semi-simple, we can, for fixed i, j use this choice $B(ij, k)$ for each $k \in \mathcal{O}(\mathcal{C})$, give a direct sum decomposition of ij. In other words, we can give a decomposition of the identity on ij as:



(2.5)

Here the ϕ^t are defined below. The pairs (ϕ, ϕ^t) for a given k are (projection, inclusion) pairs for subobjects of ij isomorphic to the simple object k. Choosing the ϕ from the basis $B(ij, k)$ ensures we exhaust all k-summands of ij without linear dependence.

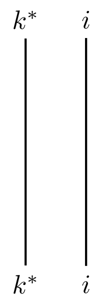
Definition 2.3. Let $\phi \in B(ij, k)$ be an element in a basis for $\mathcal{C}(ij, k)$, for simple objects i, j, k. Then a *transpose* of ϕ is the morphism ϕ^t in a dual basis for $\mathcal{C}(k, ij)$, with respect to the pairing:

$$\circ \colon \mathcal{C}(ij, k) \otimes \mathcal{C}(k, ij) \to \mathcal{C}(k, k) = \mathbb{C},$$

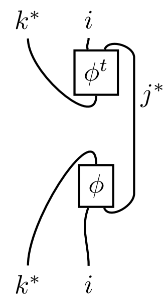
such that $\phi \circ \phi^t = \mathrm{id}_k$ and $\psi \circ \phi^t = 0$ for $\psi \in B(ij,k) - \{\phi\}$. As this pairing is non-degenerate (composing a morphism with an arbitrary morphism can only always be zero if the morphism is zero), such a dual basis, and hence transpose always exist.

Picking resolutions of the identities on ij for a fixed $i \in \mathcal{O}(\mathcal{C})$ and all $j \in \mathcal{O}(\mathcal{C})$ induces a corresponding resolution of the identity on k^*i:

Lemma 2.4. *Pick, for a fixed $i \in \mathcal{O}(\mathcal{C})$ and all $j \in \mathcal{O}(\mathcal{C})$, a resolution of the identity on ij as in Equation* (2.5)*. Then, for all $k \in \mathcal{O}(\mathcal{C})$:*

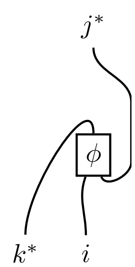


$$= \sum_{j \in \mathcal{O}(\mathcal{C})} \sum_{\phi \in B(ij,k)} \frac{d_j}{d_k}$$

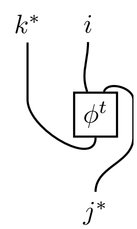


. (2.6)

Proof. We claim that we can give a direct sum decomposition of k^*i, by using for each j and $\phi \in B(ij,k)$:



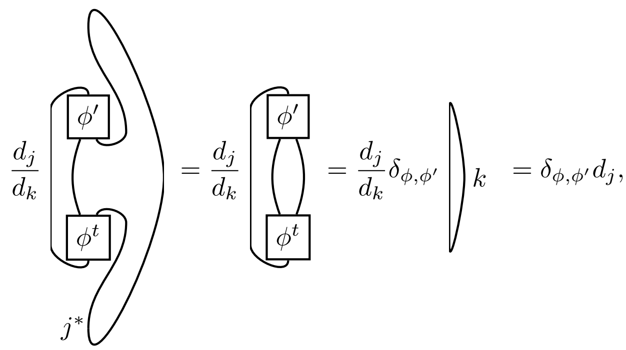
and $\frac{d_j}{d_k}$



(2.7)

as projection and inclusion to j^*, respectively. To see this, we check that composing a ϕ' and a ϕ^t along k^*i indeed gives the identity on j^* if and only if

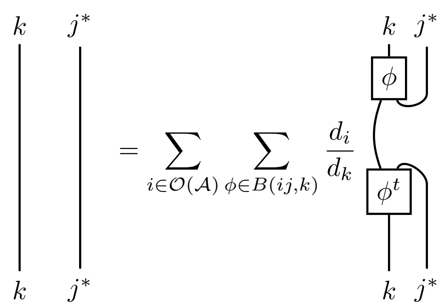
$\phi = \phi'$:



where in the first identity is just manipulation of the strings, and in the second equality we used that composing ϕ' and ϕ^t along ij gives the identity on k if $\phi = \phi'$ and zero otherwise, by Definition 2.3. As this is the trace of an endomorphism of j^* and j^* is simple, this shows that ϕ' and ϕ^t compose to the identity on j^* if and only if $\phi = \phi'$. This shows the morphisms from Equation 2.7 indeed form a linearly independent set of (projection,inclusion) pairs for each j^*. As j^* indexes through all isomorphism classes of simple objects in $\mathcal{C}$, this gives a direct sum decomposition of $k^* i$. $\square$

Similarly, we have:

Lemma 2.5. *Pick, for fixed j and all i in $\mathcal{O}(\mathcal{A})$ a resolution of the identity as in Equation* (2.5)*. Then:*

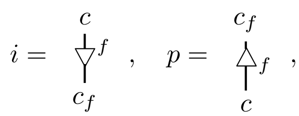


Proof. The proof is analogous to the proof of the previous lemma. $\square$

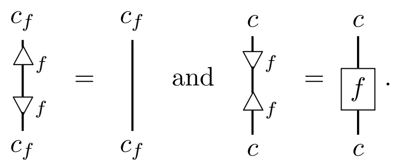
2.2.3 Idempotents and subobjects

Let $\mathcal{C}$ again be a ribbon fusion category, so it is in particular an idempotent complete category. That is, for every $c \in \mathcal{C}$ and $f \in \mathrm{End}(c)$ such that $f^2 = f$

there exists $c_f \in \mathcal{C}$, together with $i\colon c_f \hookrightarrow c$ and $p\colon c \twoheadrightarrow c_f$ satisfying $pi = \mathrm{id}_{c_f}$ and $ip = f$. Graphically, we will express this by using:



with conditions



We will refer to the object c_f as the subobject associated to f.

The following lemma will be useful later on:

Lemma 2.6. *Let c, c' be objects in an idempotent complete category $\mathcal{C}$, and let $f\colon c \to c$ and $f'\colon c' \to c'$ be idempotents, denote their associated projections, inclusions and subobjects by (p, i, c_f) and (p', i', c'_f), respectively. Suppose that $g\colon c \to c'$ is an isomorphism such that $f' = gfg^{-1}$, then $p'gi\colon c_f \to c'_f$ is an isomorphism.*

Proof. We claim that the inverse of $p'gi$ is $pg^{-1}i'$. To see this, we compute:

$$p'gipg^{-1}i' = p'gfg^{-1}i' = p'f'i' = p'i'p'i' = \mathrm{id}_{c'_f}.$$

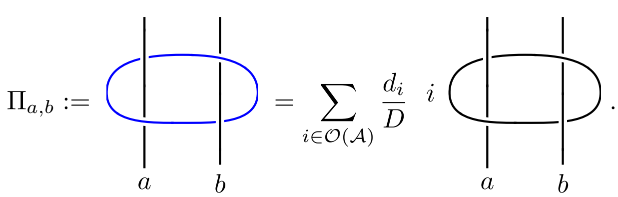
The other composite is similarly seen to be the identity. $\square$

2.3 The symmetric tensor product

2.3.1 A useful idempotent

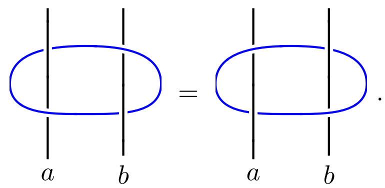
Definition of the idempotent

Recall that $\mathcal{A}$ is a symmetric ribbon fusion category. Let $a, b \in \mathcal{Z}(\mathcal{A})$. In defining the symmetric tensor product, we will use the following idempotent to pick out a subobject:

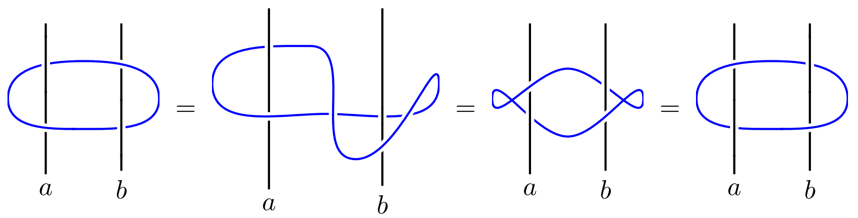


Observe that, because we are only summing over objects $i \in \mathcal{A}$, we have:

Lemma 2.7. *The morphism $\Pi_{a,b}$ can for all $a, b \in \mathcal{Z}(\mathcal{A})$ be written as*



Proof. We compute:



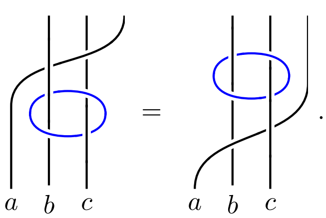
where we used that the crossing of the loop with itself corresponds to using the symmetry in $\mathcal{A}$, that the self-crossings give rise to a twist (see Equation (2.3)), and that the twist squares to 1 in a symmetric ribbon fusion category, see the discussion below Equation 2.3. □

We claim that $\Pi_{a,b}$ is an idempotent, we will prove this below, it will be a Corollary, Lemma 2.9, of another property, Lemma 2.8, we examine first.

Cloaking

The idempotent $\Pi_{a,b}$ has a very useful property, a phenomenon called *cloaking*. This lemma is a corollary of [BDSPV15, Lemma 7.1][1]. We reprove it here for convenience of the reader.

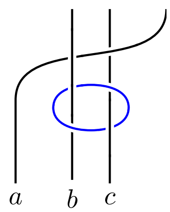
Lemma 2.8. *Let $b, c \in \mathcal{Z}(\mathcal{A})$ and $a \in \mathcal{A}$. Then the following identity holds:*



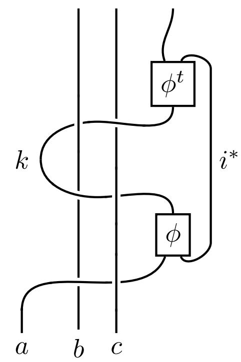
Proof. For each summand i of the loop, we decompose the identity on ai, like in Equation (2.5). Inserting this resolution of the identity at the leftmost part

[1]In the paper [BDSPV15], cloaking is phrased as taking place within a solid torus with an incoming and outgoing boundary component. To get from this result to the one here, imagine thickening the ring to a solid torus, giving the torus a boundary on each side, and passing the a strand through it.

of the loop, and pushing the morphisms along the loop to the other side, we obtain:



$$= \sum_{i,k\in\mathcal{O}(\mathcal{A})} \sum_{\phi\in B(ai,k)} \frac{t_i}{D} \tag{2.8}$$



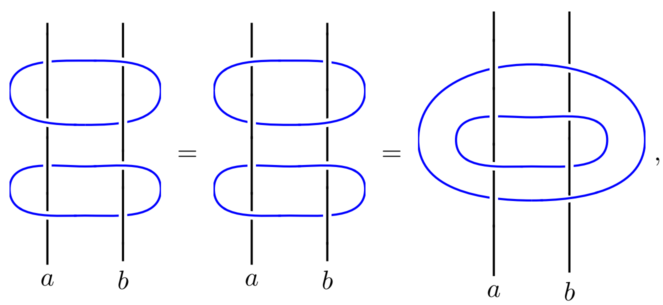
using Equation (2.6) on the rightmost part of this diagram now proves the lemma. ☐

Verifying idempotency

We still need to check $\Pi_{a,b}$ is idempotent.

Lemma 2.9. *The morphism $\Pi_{a,b}$ in $\mathcal{Z}(\mathcal{A})$ is an idempotent of $a \otimes b$.*

Proof. We compute

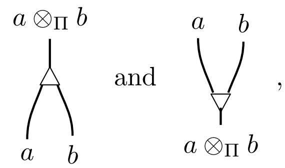


where we used Lemma 2.7 in the first step and the cloaking from Lemma 2.8 in the second. Now, we use that the loops are transparent (see Definition 2.1) to each other, as they are sums over objects of $\mathcal{A} \subset \mathcal{Z}(\mathcal{A})$. This allows us to pull the larger loop out towards the right of the diagram. This loop then evaluates to 1, leaving us with the string diagram representation of $\Pi_{a,b}$. This finishes the proof. ☐

The associated subobject

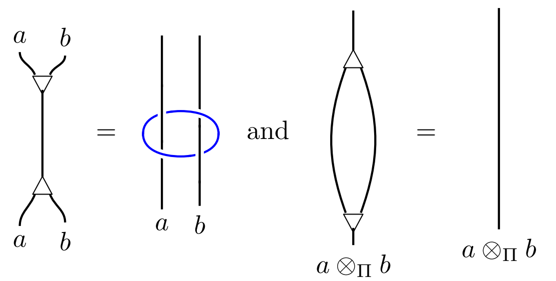
Given $a, b \in \mathcal{Z}(\mathcal{A})$, the idempotent $\Pi_{a,b}$ from Lemma 2.9 has an associated subobject denoted $a \otimes_\Pi b \in \mathcal{Z}(\mathcal{A})$. Using the notation introduced in Section

2.2.3, we introduce:



(2.9)

satisfying

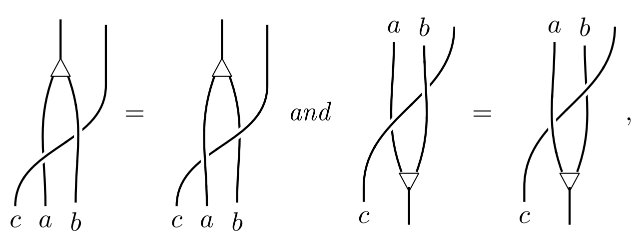


(2.10)

We have suppressed the labelling of the triangles by the idempotent $\Pi_{a,b}$, and will henceforth use unlabelled triangles to denote the inclusions and projections for $\Pi_{a,b}$.

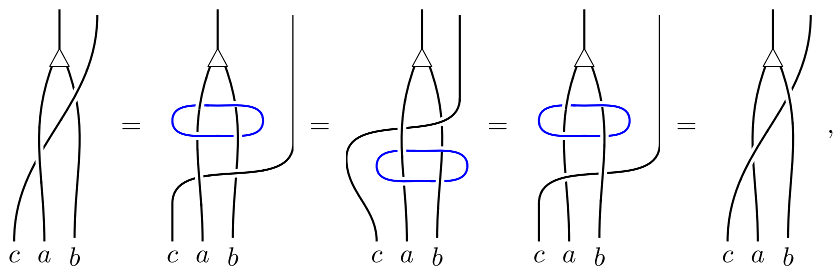
The subobject associated to $\Pi_{a,b}$ has the crucial property that the half-braidings associated to both factors agree, as is expressed by the following lemma.

Lemma 2.10. *Let $a, b \in \mathcal{Z}(\mathcal{A})$, then we have, with the notation as above:*



for any $c \in \mathcal{A}$.

Proof. We prove one of the relations, the other is similar. Using both the conditions (Equation 2.10) on the projection and inclusion in the first identity, we see that:



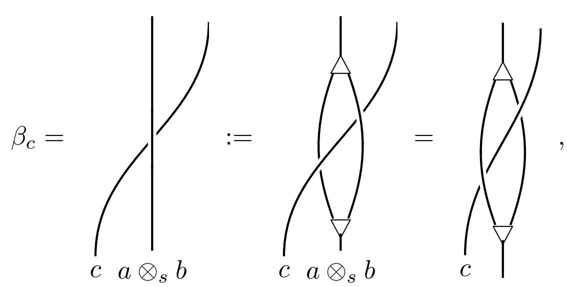
(2.11)

using the fact that the loop is transparent to the c strand in the second identity as they are both labelled by objects of $\mathcal{A}$, and the cloaking from Lemma 2.8 in the third equality. □

2.3.2 The symmetric tensor product on objects

Definition on objects

Definition 2.11. Let $a, b \in \mathcal{Z}(\mathcal{A})$, and write $\Phi\colon \mathcal{Z}(\mathcal{A}) \to \mathcal{A}$ for the forgetful functor (cf. Definition A.36). The *symmetric tensor product* $a \otimes_s b \in \mathcal{Z}(\mathcal{A})$ of a and b is the object $(\Phi(a \otimes_\Pi b), \beta)$, where $a \otimes_\Pi b$ is the subobject associated to $\Pi_{a,b}$, and β is the half-braiding with components, for $c \in \mathcal{A}$:

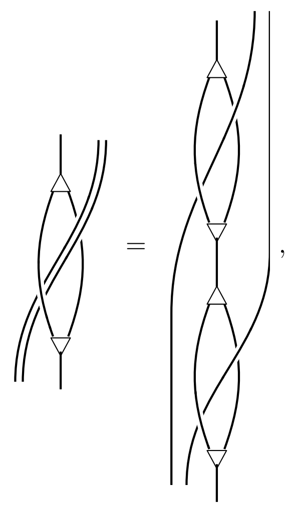


(2.12)

where the equality is a consequence of Lemma 2.10.

We observe that the β_c indeed satisfy the hexagon equation (see Definition

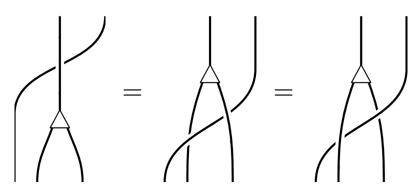
A.36 of the Drinfeld centre), which in this case reads $\beta_{cc'} = (\beta_c \otimes \mathrm{id}_{c'}) \circ (\mathrm{id}_c \otimes \beta_{c'})$:



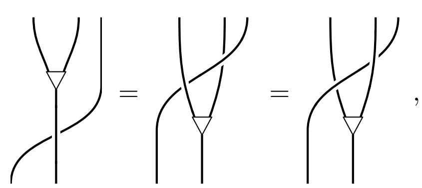
using Equation 2.10 and cloaking (Lemma 2.8).

Lemma 2.10 ensures this definition does not depend on a choice between a and b. It should be noted that that the inclusion and projection for $\Pi_{a,b}$ do not commute with the braiding, instead we have the following relation that we will call *slicing*.

Lemma 2.12 (Slicing)**.** *The half-braiding on $a \otimes_s b$ and the inclusion and projection maps for $\Pi_{a,b}$ interact as follows:*



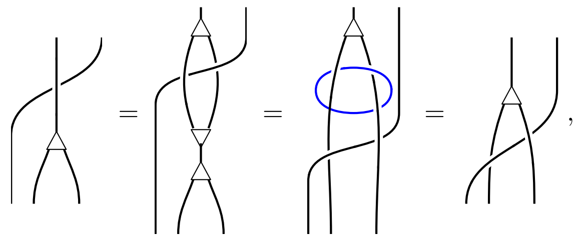
and



where the diagonal strand is labelled by an object of $\mathcal{A}$.

Proof. From the definition of the half-braiding Equation (2.12), we have, like in

Equation (2.11):

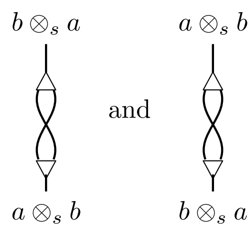


where we made use of cloaking and the properties from (2.9). The proofs of the rest of the identities are similar. □

Symmetry of the symmetric tensor product

The symmetric tensor product is indeed symmetric:

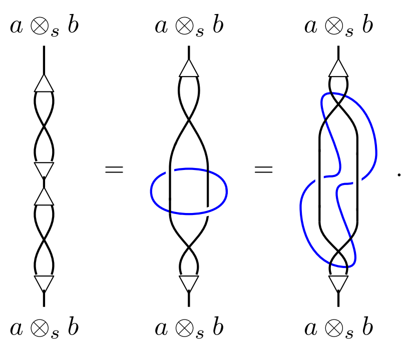
Lemma 2.13. *The symmetry in $\mathcal{A}$ induces an isomorphism between $a \otimes_s b$ and $b \otimes_s a$. That is, using the triangle notation for the inclusions and projections,*



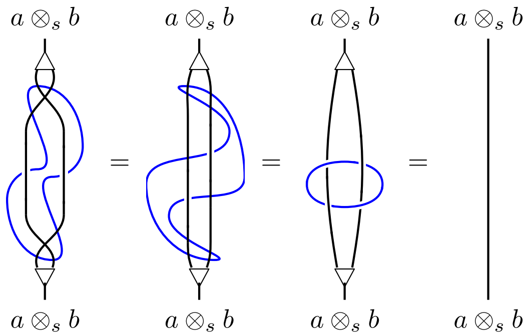
(2.13)

are mutually inverse morphisms in $\mathcal{Z}(\mathcal{A})$.

Proof. We will first establish that the symmetry morphisms are mutually inverse in $\mathcal{A}$, then we will prove they lift to morphisms in $\mathcal{Z}(\mathcal{A})$. Consider the composite

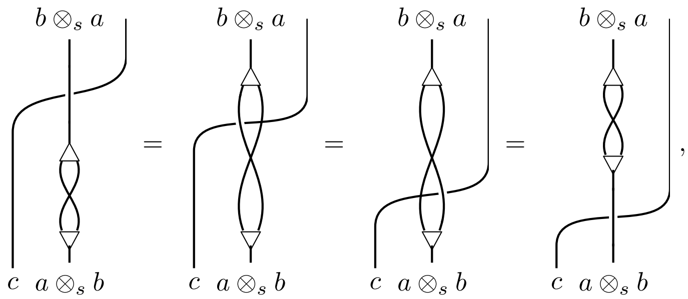


Here the unresolved crossings denote the symmetry in $\mathcal{A}$. The first step comes from replacing the inclusion followed by the projection with the idempotent (cf. Section 2.2.3). The second uses the fact that the symmetry in $\mathcal{A}$ allows us to do Reidemeister moves which involve only the unresolved crossings. We can now swap the strands with the braiding morphisms for a and b, undoing the symmetry crossings between the a and b strands, and get:



A similar argument shows the other composite is also the identity.

We still need to establish that the morphisms are indeed morphisms in $\mathcal{Z}(\mathcal{A})$. That is, we need to show that they commute with the braiding as defined in Equation (2.12). We compute, using Lemma 2.12:

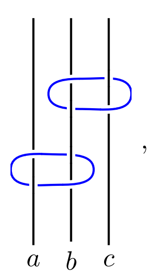


as desired. □

Associativity

Before we discuss the associators, it is helpful to examine what at a triple product $(a \otimes_s b) \otimes_s c$ looks like.

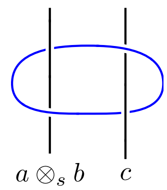
Lemma 2.14. *The triple products* $(a \otimes_s b) \otimes_s c$ *and* $a \otimes_s (b \otimes_s c)$ *have as underlying object the subobject associated to the idempotent*



, (2.14)

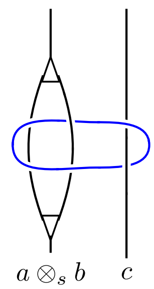
interpreted as endomorphism of $(ab)c$ *and* $a(bc)$*, respectively, using the (suppressed) associators.*

Proof. By definition, the underlying object of $(a \otimes_s b) \otimes_s c$ is the subobject associated to the idempotent



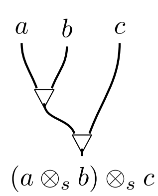
,

where the overcrossing on the strand $a \otimes_s b$ corresponds to Equation (2.12). Spelling this out, we get:

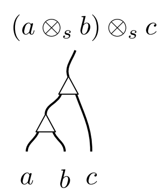


.

We now claim that

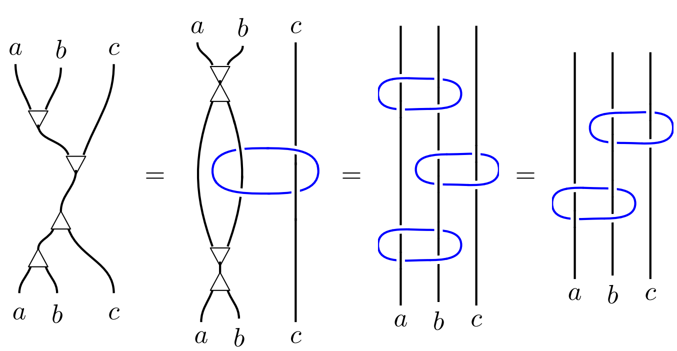


and



,

exhibit $(a \otimes_s b) \otimes_s c$ as the subobject associated to the idempotent from Equation (2.14). From the properties of the inclusions and projections involved, we see that the composition along abc indeed is the identity. Composing along $(a \otimes_s b) \otimes_s c$, we get:

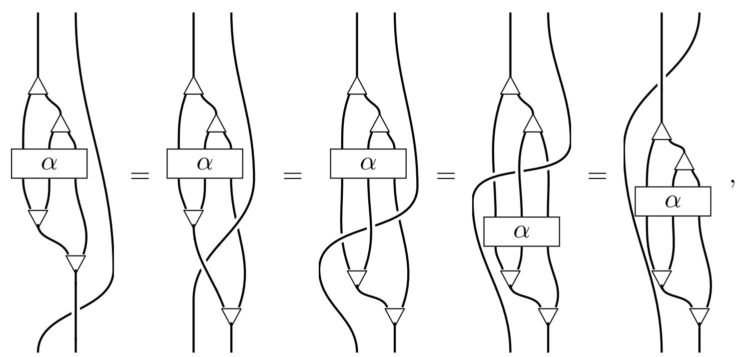


, (2.15)

where in the last step we used that the rings are transparent to each other and idempotent, the first two steps come from combining inclusion and projections to idempotents. The argument for $a \otimes_s (b \otimes_s c)$ is analogous. □

Lemma 2.15. *The associators of $\mathcal{A}$ induce isomorphisms between $(a \otimes_s b) \otimes_s c$ and $a \otimes_s (b \otimes_s c)$ for all $a, b, c \in \mathcal{Z}(\mathcal{A})$.*

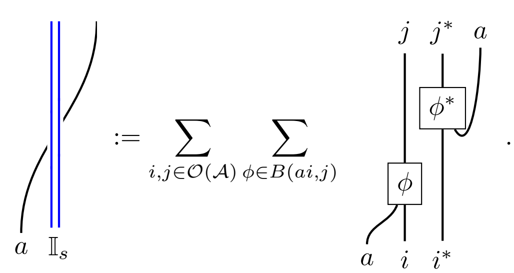
Proof. From Lemma 2.14, we know that that the triple products have underlying objects that are the subobjects associated to idempotents that are conjugate to each other along the associators $\alpha\colon (ab)c \to a(bc)$. This means we are in the situation of Lemma 2.6 and the associators will induce isomorphisms between these subobjects. We still have show that these isomorphisms are compatible with the half-braidings, i.e. that the induced morphisms are indeed in $\mathcal{Z}(\mathcal{A})$. To do this, we check that, explicitly inserting the associator α for this proof:



where we made repeated use of slicing (Lemma 2.12). To pass the braiding past the associator, we have used the naturality of the braiding. □

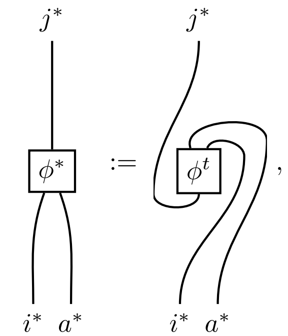
Unit

Definition 2.16. The *symmetric unit* $\mathbb{I}_s$ is the object $\sum_{i \in \mathcal{O}(\mathcal{A})} ii^*$, equipped with the half braiding:



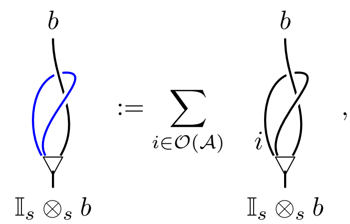
(2.16)

The double strand will henceforth be used to denote the identity on $\mathbb{I}_s$. In the above formula ϕ^* denotes



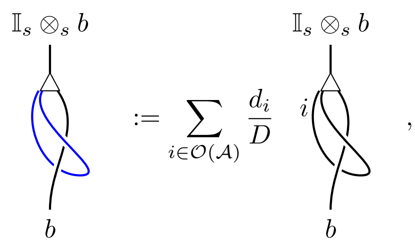
and ϕ^t was introduced in Definition 2.3.

We will show that this object acts as a monoidal unit for the symmetric tensor product together with the left unitor built from evaluation morphisms



(2.17)

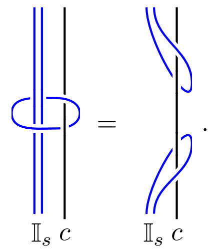
where the double strand coming out of the inclusion on the left hand side denotes the object $\mathbb{I}_s$. The right unitor is obtained by reflecting the above diagram in a vertical line. We claim, and prove below in Lemma 2.18, that the left unitor has an inverse given by:



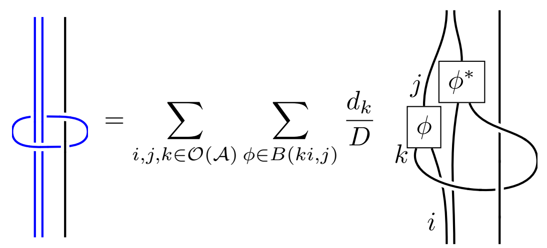
(2.18)

and the inverse for the right unitor is correspondingly given by reflecting the above diagram in a vertical line. To prove these statement and that this indeed gives the monoidal unit, we will make use of the following property we will refer to as *snapping*:

Lemma 2.17 (Snapping)**.** *For any $c \in \mathcal{Z}(\mathcal{A})$ we have:*

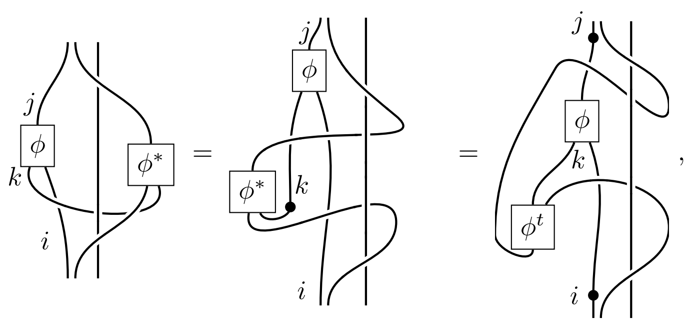


Proof. Unpacking the definition of the half-braiding on $\mathbb{I}_s$, we get:

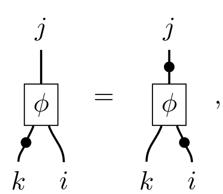


(2.19)

We can manipulate the summands on the right hand side, using Equation (2.1), to get:

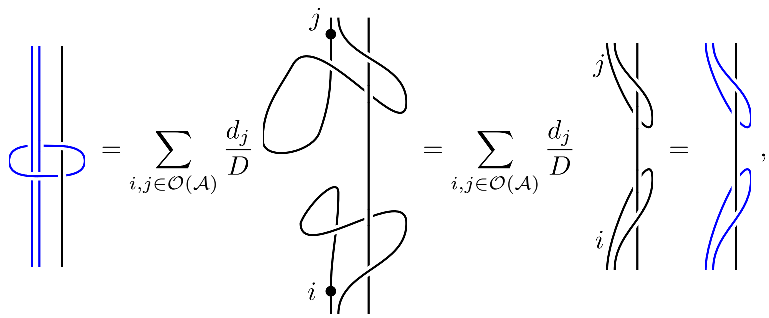


where in the first equality a self-intersection gave a twist (see Equation (2.3)) on the k strand. The third equality uses the definition of ϕ^* combined with:



(2.20)

which follows from the naturality of the twist, together with the fact that in a symmetric fusion category the twist is a monoidal automorphism of the identity functor that squares to 1. We can now apply Lemma 2.5 to obtain, performing the sum over ϕ and k,

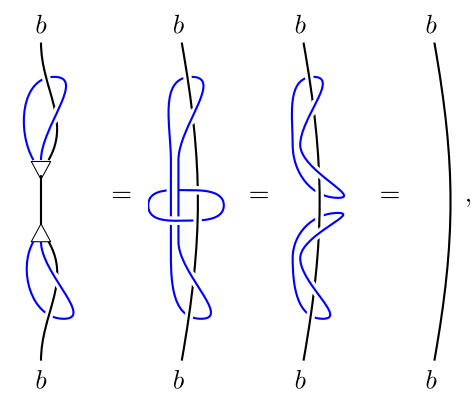


where in the second equality we cancelled twists with self-intersections. □

The object $\mathbb{I}_s$ does indeed act as a unit for the symmetric tensor product on $\mathcal{Z}(\mathcal{A})$:

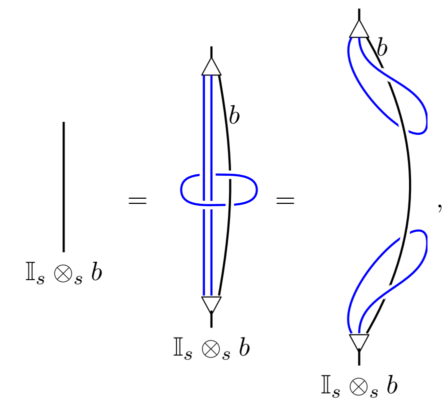
Lemma 2.18. *The symmetric tensor product of $\mathbb{I}_s$ with any object $b \in \mathcal{Z}(\mathcal{A})$ is isomorphic to b as object in $\mathcal{Z}(\mathcal{A})$, along the isomorphism given in Equation* (2.17)*. Similarly, $b \otimes_s \mathbb{I}_s \cong b$, along the isomorphisms given by reflecting the diagrams above in a vertical line.*

Proof. We first prove that the morphisms from Equations (2.17) and (2.18) are inverse to each other, and then establish they are morphisms in $\mathcal{Z}(\mathcal{A})$. Composing along $\mathbb{I}_s \otimes_s b$, we see we need to check that:



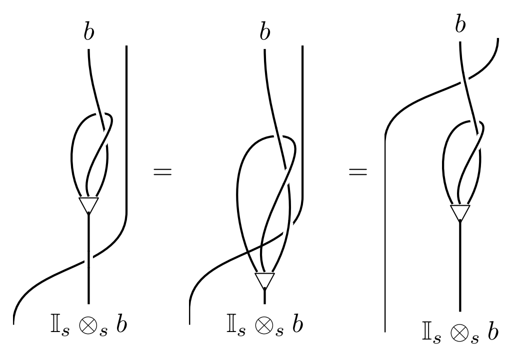
(2.21)

where we used snapping (Lemma 2.17), and that in the last steps the rings come off an evaluate to a factor 1. For the other composition, note that:



using snapping in the last step.

To see that the morphisms are indeed morphisms in $\mathcal{Z}(\mathcal{A})$, we check that:



where the first step is Lemma 2.12, and the second step uses Equation (2.2). The proof that $b \otimes_s \mathbb{I}_s \cong b$ along the specified isomorphisms is analogous. $\square$

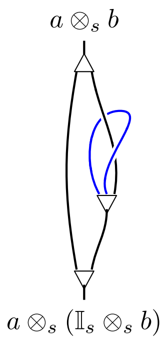
For $\mathbb{I}_s$ to be a unit for the symmetric tensor product, the isomorphisms from Lemma 2.18 need to satisfy the triangle equality, that is:

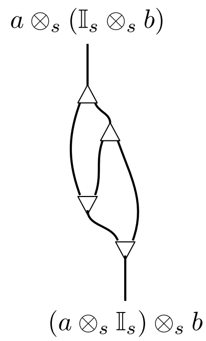
$$\begin{tikzcd}
(a \otimes_s \mathbb{I}_s) \otimes_s b \arrow[rr] \arrow[dr] & & a \otimes_s (\mathbb{I}_s \otimes_s b) \arrow[dl] \\
& a \otimes_s b &
\end{tikzcd} .$$

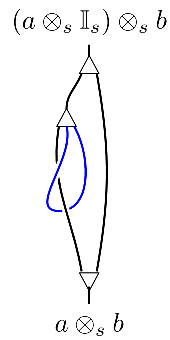
commutes for all $a, b \in \mathcal{Z}(\mathcal{A})$, where the downwards maps are the unitor isomorphism and the top is the associator.

Lemma 2.19. *The isomorphisms from Lemma 2.18 satisfy the triangle equality.*

Proof. We will show that the clockwise composite $a \otimes_s b \to (a \otimes_s \mathbb{I}_s) \otimes_s b \to a \otimes_s (\mathbb{I}_s \otimes_s b) \to a \otimes_s b$ is the identity on $a \otimes_s b$. That is, we are considering the composite of



,


and


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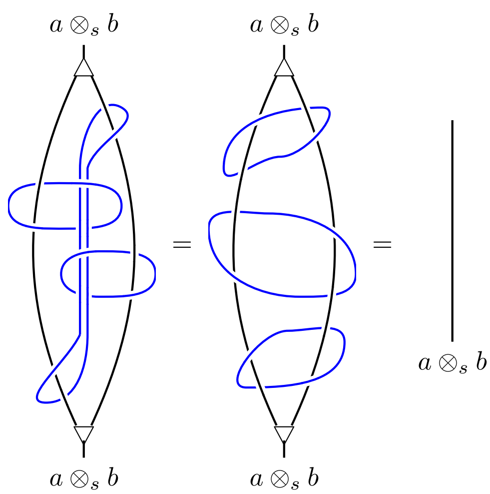
When composing, we encounter Equation (2.15) and its mirror image. Plugging this in right away and remembering the rings are idempotent, we get



.

Here the first equality is an application of snapping to the two horizontal rings (Lemma 2.17), the second uses the fact that the rings cancel with the inclusion and projection morphisms. □

2.3.3 The symmetric tensor product as a functor

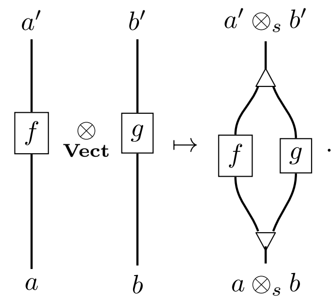
We have so-far given objectwise definitions of the ingredients needed to define the symmetric tensor product. In this section we will combine these definitions to make the symmetric tensor product into a monoidal structure. The final ingredient needed is a definition of the symmetric tensor product on morphisms.

Definition on morphisms

Definition 2.20. The *symmetric tensor product*

$$\otimes_s \colon \mathcal{Z}(\mathcal{A}) \boxtimes \mathcal{Z}(\mathcal{A}) \to \mathcal{Z}(\mathcal{A})$$

is a symmetric monoidal structure on $\mathcal{Z}(\mathcal{A})$ defined on objects in Definition 2.11. On morphisms $f\colon a \to a', g\colon b \to b'$, it is given by

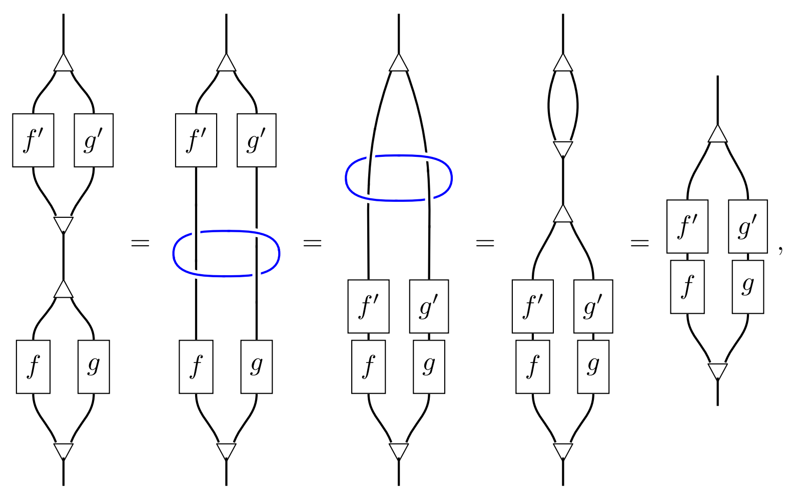


(2.22)

The unit for this monoidal structure is given in Definition 2.16. The associators are induced by the associators of $\mathcal{A}$ as described in Lemma 2.15. The symmetry is induced by the symmetry morphisms in $\mathcal{A}$, as described in Lemma 2.13.

Lemma 2.21. *The prescription from Definition 2.20 is a functor.*

Proof. Observe that we have, for f, f' and g, g' morphisms in $\mathcal{Z}(\mathcal{A})$:



where in the second step we used naturality of the braiding in $\mathcal{Z}(\mathcal{A})$. $\square$

The symmetric tensor product as symmetric monoidal structure

Now that we have promoted $\otimes_s$ to a functor, it makes sense to ask whether it defines a symmetric tensor product on $\mathcal{Z}(\mathcal{A})$.

To see $\otimes_s$ is weakly associative, note that we have shown that the maps induced from the associators of $\mathcal{A}$ give isomorphisms between the two possibilities for the triple product (Lemma 2.15). As the associators for $\mathcal{A}$ satisfy the pentagon equations, so will the induced maps. Furthermore, an argument analogous to the proof of functoriality will establish that these isomorphisms are natural.

For weak unitality, observe that, in Lemmas 2.18 and 2.19, we have established $\mathbb{I}_s$ as the unit for $\otimes_s$.

To establish symmetry of $\otimes_s$, we recall that we have shown that the symmetry in $\mathcal{A}$ induces isomorphisms between the swapped orders of taking the symmetric tensor product (Lemma 2.13). These induced morphisms will give a natural transformation that satisfies the hexagon equations.

Collecting these observations, we have therefore shown that:

Theorem 2.22. *$(\mathcal{Z}(\mathcal{A}), \otimes_s, \mathbb{I}_s)$ is a symmetric monoidal category.*

The forgeful functor is lax monoidal

The goal of this section is to establish:

Proposition 2.23. *The forgetful functor $\Phi\colon (\mathcal{Z}(\mathcal{A}), \otimes_s) \to (\mathcal{A}, \otimes_{\mathcal{A}})$ is symmetric lax monoidal (see Definition A.22).*

Proof. We have to provide a natural transformation λ from $\Phi(-)\Phi(-)$ to $\Phi(-\otimes_s -)$, so for each $c = (a, \beta)$ and $c' = (a', \beta')$ a map:

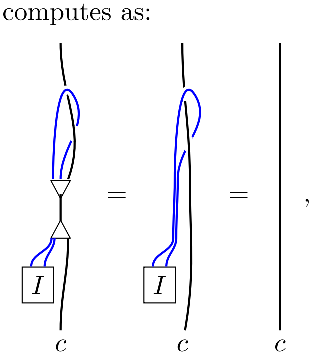
$$\mu_{c,c'}\colon aa' \to \Phi(c \otimes_s c') = \Phi(c \otimes_\Pi c').$$

We claim that the image under Φ of the projection $p_{c,c'}$ associated to $\Pi_{c,c'}$ will work. First of all, we observe that the forgetful functor is strictly monoidal for $\otimes_c$ (as the monoidal structure on the Drinfeld centre is defined in terms of the one on $\mathcal{A}$), so the image of $\Phi(p_{c,c'}) = \mu_{c,c'}$ is certainly a map between aa' and $\Phi(c \otimes_\Pi c')$. As the associators are defined using the projection $p_{c,c'}$, this map is automatically compatible with the associators, c.f. the first diagram in Definition A.22.

Next, we need to provide a map

$$\mu_0\colon \mathbb{I}_{\mathcal{A}} \to \Phi(\mathbb{I}_s) = \bigoplus_{i \in \mathcal{O}(\mathcal{A})} ii^*,$$

we take this to be D times the inclusion I of $\mathbb{I} \cong \mathbb{III}$ into $\mathbb{I}_s$. Tracing trough the second compatibility diagram in Definition A.22, the composite along the right hand side of the diagram computes as:



where the first step uses snapping (the resulting ring on the top comes off immediately), and the second step comes from the observation that the unit braids trivially with all other objects.

Finally, we have to show that the symmetry for $\otimes_s$ is sent to the symmetry in $\mathcal{A}$, but this follows directly from its definition in Lemma 2.13. $\square$

Basic Properties of the Symmetric Tensor Product

In this section we will do some basic computations with the symmetric tensor product, that tell us how the symmetric tensor product behaves with respect to the subcategory $\mathcal{A} \subset \mathcal{Z}(\mathcal{A})$. We start with:

Lemma 2.24. *Let a and a' be objects of $\mathcal{A} \subset \mathcal{Z}(\mathcal{A})$. Then*

$$a \otimes_s a' = a \otimes_c a'.$$

Proof. The object $a \otimes_s a'$ is defined in terms of the suboject associated to $\Pi_{a,a'} : a \otimes_c a' \to a \otimes_c a'$. As the objects of $\mathcal{A}$ are transparent to each other, we see that $\Pi_{a,a'} = \mathrm{id}_{a \otimes_c a'}$, and the result follows. $\square$

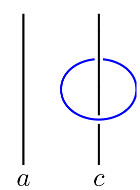
The subcategory $\mathcal{A}$ is orthogonal to the rest of $\mathcal{Z}(\mathcal{A})$, in the sense that:

Lemma 2.25. *Let $a \in \mathcal{A}$ and let $c \in \mathcal{Z}(\mathcal{A})$ be a simple object not in $\mathcal{A}$. Then:*

$$a \otimes_s c = 0,$$

where 0 denotes the zero object of $\mathcal{Z}(\mathcal{A})$.

Proof. Recall that $a \otimes_s c$ is defined using the subobject associated to the idempotent $\Pi_{a,c}$ from Lemma 2.9, and is therefore zero if and only if the idempotent is zero on $a \otimes_c c$. Using that the objects of $\mathcal{A}$ are transparent with respect to each other we see that the idempotent computes as:

$\Pi_{a,c} =$ 

,

which is zero if endomorphism of the simple c defined by the right part of the diagram is zero. This in turn happens if and only if the trace of this endomorphism is zero. Its trace is by definition:

$$\sum_{i \in \mathcal{O}(\mathcal{A})} \frac{d_i}{D} S(c,i),$$

where $S(c,i)$ is the S-matrix entry (see [Müg03b]) for c and i. By [Müg03b, Lemma 2.13], this trace computes as:

$$\sum_{i \in \mathcal{O}(\mathcal{A})} \frac{d_i}{D} S(c,i) = d_{c'} \chi_{\mathcal{Z}_2(\mathcal{A},\mathcal{Z}(\mathcal{A}))}(c).$$

Here $\chi_{\mathcal{Z}_2(\mathcal{A},\mathcal{Z}(\mathcal{A}))}$ denotes the characteristic function on the objects of $\mathcal{Z}_2(\mathcal{A},\mathcal{Z}(\mathcal{A}))$. The centraliser of $\mathcal{A}$ in its Drinfeld centre is $\mathcal{A}$, so all in all we see that $\Pi_{a,c} = 0$ if c is not in $\mathcal{A}$. Hence $a \otimes_s c$ is zero for all simple c not in $\mathcal{A}$. □

These two results combine to give:

Proposition 2.26. *Let $a \in \mathcal{A}$ and $c \in \mathcal{C}$. Denote by $c_{\mathcal{A}}$ the maximal summand of c that is an object of $\mathcal{A}$. Then:*

$$a \otimes_s c = a \otimes_c c_{\mathcal{A}}.$$

Note that this also implies, by associativity and symmetry of the symmetric tensor product, that $c \otimes_s d$ is never an object of $\mathcal{A}$ if c and $\lceil$ have no summands in $\mathcal{A}$.

We can also prove the following relationship between the symmetric unit and the convolution tensor product between objects of a and objects of $\mathcal{Z}(\mathcal{A})$.

Lemma 2.27. *Let $a \in \mathcal{A}$ and $z \in \mathcal{Z}(\mathcal{A})$. Then:*

$$a \otimes_c z \cong (a \otimes_c \mathbb{I}_s) \otimes_s z.$$

Proof. The object $(a \otimes_c \mathbb{I}_s) \otimes_s z$ is computed in terms of the idempotent $\Pi_{a\otimes_c \mathbb{I}_s, z}$. As a is transparent, we have that:

$$\Pi_{a\otimes_c \mathbb{I}_s, z} = \mathrm{id}_a \otimes_c \Pi_{\mathbb{I}_s, z} = \mathrm{id}_a \otimes_c \mathrm{id}_z,$$

where in the second step we used that $\mathbb{I}_s$ is the unit for $\otimes_s$. The result now follows. □

2.4 The symmetric tensor product under Tannaka duality

Any symmetric fusion category is, by Tannaka Duality (Theorem A.35), equivalent to the representation category of a finite (super-)group. Furthermore, as discussed in Section A.2.3, the Drinfeld centre of such a representation category can be viewed as the category of equivariant vector bundles over (the underlying group of) this (super-)group (Definition A.37). This category admits two obvious tensor products, the convolution tensor product (Definition A.39) and the fibrewise tensor product (Definition 2.29).

The goal of this section is to first show that for $\mathcal{A}$ Tannakian (Definition A.32), the symmetric tensor product on $\mathcal{Z}(\mathcal{A})$ translates to the fibrewise tensor product when viewing the Drinfeld centre as equivariant vector bundles. After this, we will examine what the symmetric tensor product becomes when the symmetric fusion category is super-Tannakian. We will see that in this case, the symmetric tensor product translates to a twisted version of the fibrewise tensor product that takes into account the super-group structure.

2.4.1 Tannakian Case

In this section we will examine what the symmetric tensor product on $\mathcal{Z}(\mathcal{A})$ gives in the case where $\mathcal{A} = \mathrm{Rep}(G)$, where G is a finite group. We will show that:

Theorem 2.28. *Let G be a finite group. Then the equivalence $\mathcal{E}$ from Equation* (A.16) *between $(\mathcal{Z}(\mathrm{Rep}(G)), \otimes_s)$ and $(\mathbf{Vect}_G[G], \otimes_f)$ is a symmetric monoidal equivalence. Here $\otimes_f$ denotes the fibrewise tensor product from Definition 2.29.*

The proof of this theorem will take up the rest of this section. We start by giving the definition of the fibrewise tensor product.

Definition 2.29. The *fibrewise tensor product on* $\mathbf{Vect}_G[G]$ is given by

$$(V \otimes_f W)_g = V_g \otimes W_g,$$

with G-action $\rho_V \otimes \rho_W$.

This tensor product is clearly symmetric with symmetry given by the switch map of vector spaces.

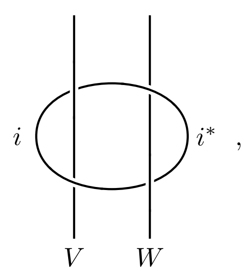
We will now examine what the idempotent $\Pi_{a,b}$ looks like in $\mathbf{Vect}_G[G]$. In particular, we will establish the following:

Lemma 2.30. *Let $V, W \in \mathbf{Vect}_G[G]$ then the idempotent $\Pi_{V,W} \colon V \otimes_c W \to V \otimes_c W$ is given by*

$$\Pi_{V,W}|_{V_{g_1} \otimes W_{g_2}} = \begin{cases} \mathrm{id} & \text{for } g_1 = g_2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By definition, $\Pi_{V,W}$ is given by

$$\sum_{i \in \mathrm{IrRep(G)}} \frac{d_i}{D}$$



where we put the label i^* to emphasise the object going up is i^*. Recall, from Section A.2.3 that we are viewing $i \in \mathrm{Rep}(G)$ as an object in $\mathbf{Vect}_G[G]$ by regarding it as a vector bundle supported by $[e]$. The convolution tensor product (Definition A.39) between any bundle E and a bundle $F = F_e$ supported by $[e]$ has fibres given by

$$(E \otimes F)_g = E_g \otimes F_e.$$

We claim that $\Pi_{V,W}$ acts as a sum of endomorphisms of the summands $V_{g_1} \otimes W_{g_2}$ of the fibres over $g = g_1 g_2$ of $V \otimes_c W$. The braidings on the V and W

strands with i and i^* will individually fibrewise be automorphisms of $V_{g_1} \otimes i$ and $i^* \otimes W_{g_2}$. Precomposing with co-evaluation and postcomposing with evaluation for i combines these to automorphisms of $V_{g_1} \otimes W_{g_2}$, for each i in the sum. This means the idempotent will be a direct sum of maps

$$\Pi_{V_{g_1}, W_{g_2}} : V_{g_1} \otimes W_{g_2} \to V_{g_1} \otimes W_{g_2},$$

for each possible combination of fibres V_{g_1} and W_{g_2}.

We now want to compute what these automorphisms are. By the definition of the braiding (Definition A.40), each of these maps $\Pi_{V_{g_1}, W_{g_2}}$ is given by the composite of the evaluation and coevaluation for i with, denoting by $\rho^i(g)$ the action of G on the representation i,

$$V_{g_1} \otimes i \otimes i^* \otimes W_{g_2} \xrightarrow{\mathrm{id}_V \otimes \mathrm{id}_i \otimes \rho^{i^*}(g_2) \otimes \mathrm{id}_W} \otimes V_{g_1} \otimes i \otimes i^* \otimes W_{g_2}$$

and

$$V_{g_1} \otimes i \otimes i^* \otimes W_{g_2} \xrightarrow{\mathrm{id}_V \otimes \rho^{i}(g_1) \otimes \mathrm{id}_i \otimes \mathrm{id}_W} V_{g_1} \otimes i \otimes i^* \otimes W_{g_2},$$

where we have gotten rid of unnecessary switch maps between vector spaces. By unitarity of the representations, $\mathrm{ev} \circ \mathrm{id}_i \otimes \rho^{i^*}(g_2) = \mathrm{ev} \circ \rho^i(g_2^{-1}) \otimes \mathrm{id}_{i^*}$. The evaluation and coevaluation combine to a trace, so we see that

$$\Pi_{V_{g_1}, W_{g_2}} = \sum_{i \in \mathrm{IrRep}(G)} \frac{d_i}{D} \mathrm{tr}(\rho^i(g_2^{-1})\rho^i(g_1)) = \sum_{i \in \mathrm{IrRep}(G)} \frac{d_i}{D} \chi_i(g_2^{-1} g_1),$$

where χ_i denotes the character of i. We recognise the right hand side as $\frac{1}{D}$ times the character of the group algebra, viewed as a representation of G, evaluated on $g_2^{-1} g_1$. As the group acts freely on itself, this character is D times the characteristic function of the conjugacy class of the identity element. This proves the lemma. □

Corollary 2.31. *The subobject associated to $\Pi_{V,W}$ has fibres*

$$(V \otimes_\Pi W)_{g'} = \bigoplus_{g^2 = g'} V_g \otimes W_g. \tag{2.23}$$

To compare the symmetric tensor product to the fibrewise product, we need to see what effect equipping this object with the half-braiding from Equation (2.12) has. We claim that this replaces g^2 by g. This will establish:

Lemma 2.32. *For any $V, W \in \mathbf{Vect}_G[G]$,*

$$\mathcal{E}(V \otimes_s W) = V \otimes_f W.$$

Proof. Unpacking the definition of the half-braiding, we see that the braiding on $V \otimes_s W$ with respect to $a \in \mathrm{Rep}(G)$ is given by, on each summand in Equation (2.23),

$$aV_g W_g \xrightarrow{s_{V_g, a} \otimes \mathrm{id}_W \circ (\rho^a(g) \otimes \mathrm{id}_{V \otimes W})} V_g a W_g \xrightarrow{s_{W_g, a}} V_g W_g a,$$

where the first map is the braiding from Equation (A.40) and the second the symmetry in $\mathcal{A}$. By monoidality of the symmetry s, this composite is the same as:

$$aV_gW_g \xrightarrow{s_{a,V_gW_g}\circ(\rho^a(g)\otimes \mathrm{id}_{V\otimes W})} V_gW_ga.$$

Comparing this with Definition (A.40), this is saying that $V \otimes_s W$ is the bundle with fibres

$$(V \otimes_s W)_g = V_g \otimes W_g,$$

and this is what we wanted to show. □

Combining Corollary 2.31 and Lemma 2.32 now proves Theorem 2.28.

2.4.2 Super-Tannakian Case

We will now examine the case where $\mathcal{A}$ is super-Tannakian (Definition A.32) and hence, by Deligne's Theorem A.35, equivalent to $\mathrm{Rep}(G,\omega)$ for some finite super-group (Definition A.33). The structure of the Drinfeld centre in this case is discussed in Section A.2.3. The Drinfeld centre is still $\mathbf{Vect}_G[G]$. However, the inclusion of $\mathrm{Rep}(G,\omega)$ into $\mathbf{Vect}_G[G]$ will be different, and the symmetric tensor product gives rise to a new tensor product on $\mathbf{Vect}_G[G]$.

Definition 2.33. Let (G,ω) be a finite super-group. The *fibrewise super-tensor product* of homogeneous (see Definition A.47) $V,W \in \mathbf{Vect}_G[G]$ is the G-equivariant vector bundle $V \otimes_f^\omega W$ with fibres

$$(V \otimes_f^\omega W)_g = V_{\omega^{|W|}g} W_{\omega^{|V|}g},$$

and G-action given by the tensor product of the G-actions.

Remark 2.34. We can interpret Definition 2.33 as follows: for every choice of central order 2 element of a finite group G, there is a symmetric tensor product on $\mathbf{Vect}_G[G]$.

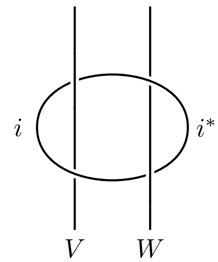
In this section, we will prove the following:

Theorem 2.35. *Let (G,ω) be a finite super-group. Then the equivalence between $(\mathcal{Z}(\mathrm{Rep}(G,\omega),\otimes_s)$ and $(\mathbf{Vect}_G[G],\otimes_f^\omega)$ is symmetric monoidal.*

The main difficulty in proving this Theorem is that, as asserted by Proposition A.45, the inclusion functor from $\mathrm{Rep}(G,\omega)$ to $\mathcal{Z}(\mathrm{Rep}(G,\omega))$ does not only hit bundles supported by $[e]$. This means we have revisit Lemma 2.30 and its proof. We will do this step by step below.

The starting point is again that $\Pi_{V,W}$ is given by

$$\sum_{i\in\mathrm{IrRep}(G,\omega)} \frac{d_i}{D} \tag{2.24}$$



Recall (see Definition A.33), that the i are either even or odd, and that (Proposition A.45) even representations are viewed as bundles supported by $[e]$, while odd representations are viewed as bundles supported by $[\omega]$.

Each even i summand in Equation (2.24) will, just as in the Tannakian case, contribute an automorphism of each $V_{g_1} \otimes W_{g_2}$ given by multiplication by $\chi_i(g_2^{-1}g_1)$, regardless of the parity of V and W.

Now suppose that i is odd. Since ω is the only element its conjugacy class, analogous reasoning to the one applied in the Tannakian case tells us that for such odd i we get an endomorphism of $V_{g_1} \otimes W_{g_2}$, let us denote it by

$$\Pi^i_{V_{g_1},W_{g_2}} : V_{g_1} \otimes W_{g_2} \to V_{g_1} \otimes W_{g_2}.$$

We now want to compute what this map is. It is given by the composite of the appropriate evaluation and coevaluation with

$$V_{g_1} \otimes i \otimes i^* \otimes W_{g_2} \xrightarrow{(-1)^{|V|}\mathrm{id}_V \otimes \mathrm{id}_i \otimes \rho^{i^*}(g_2) \otimes \mathrm{id}_W} V_{g_1} \otimes i \otimes i^* \otimes W_{g_2}$$

and

$$V_{g_1} \otimes i \otimes i^* \otimes W_{g_2} \xrightarrow{(-1)^{|W|}\mathrm{id}_V \otimes \rho^{i}(g_1) \otimes \mathrm{id}_i \otimes \mathrm{id}_W} V_{g_1} \otimes i \otimes i^* \otimes W_{g_2},$$

where $|V|, |W| \in \{0, 1\}$ denote the parity of V and W, as by restricting to simple objects we can assume V and W to be homogeneous, see Lemma A.46. The signs come from the braiding between V and i and W and i^*, respectively. From here, we can apply the same arguments as in the Tannakian case to arrive at:

$$\Pi_{V_{g_1},W_{g_2}} = \sum_{i\in\mathrm{IrRep}_0(G,\omega)} \frac{d_i}{D}\mathrm{tr}(\rho^i(g_2^{-1}g_1)) + \sum_{i\in\mathrm{IrRep}_1(G,\omega)} \frac{d_i}{D}(-1)^{|V|+|W|}\mathrm{tr}(\rho^i(g_2^{-1}g_1)),$$

where we have denoted sets of representatives of the even and odd simple objects of $\mathrm{Rep}(G, \omega)$ by $\mathrm{IrRep}_0(G, \omega)$ and $\mathrm{IrRep}_1(G, \omega)$, respectively. Now, recall that, by definition, ω acts as id on even and as $-$id on odd i. This means that

$$\chi_i(\omega^{|V|+|W|}g_2^{-1}g_1) = \begin{cases} \chi_i(g_2^{-1}g_1) & \text{for } i \text{ even} \\ (-1)^{|V|+|W|}\chi_i(g_2^{-1}g_1) & \text{for } i \text{ odd} \end{cases}.$$

We can use this to rewrite:

$$\Pi_{V_{g_1},W_{g_2}} = \sum_{i\in\mathrm{IrRep}(G)} \frac{d_i}{D}\chi_i(\omega^{|V|+|W|}g_2^{-1}g_1) = \begin{cases}\mathrm{id} & \text{for } g_2^{-1}g_1 = \omega^{|V|+|W|}\\ 0 & \text{otherwise}\end{cases},$$

which is the super version of Lemma 2.30. This means that, analogous to Corollary 2.23, we have:

Corollary 2.36. *The subobject associated to $\Pi_{V,W}$ is the equivariant vector bundle with fibres:*

$$(V\otimes_\Pi W)_{g'} = \bigoplus_{g^2=\omega^{|V|+|W|}g'} V_{\omega^{|V|+|W|}g}W_g = \bigoplus_{g^2=\omega^{|V|+|W|}g'} V_{\omega^{|W|}g}W_{\omega^{|V|}g},$$

for V and W homogeneous.

As we can decompose any vector bundle into homogeneous summands, this Corollary completely determines the subobject underlying the symmetric tensor product of any two vector bundles.

Following the exposition of the Tannakian case, our next task is now to determine what the half-braiding (Equation (2.12)) is that we will equip this object with to form the symmetric tensor product.

We will again compute what this braiding is summandwise. So, let $a\in\mathrm{Rep}(G,\omega)$ be homogeneous and $V_{\omega^{|V|+|W|}g}W_g$ be a summand in the fibre over $\omega^{|V|+|W|}g^2$. Unpacking the definition of the half-braiding, we get, analogously to the Tannakian case:

$$aV_{\omega^{|V|+|W|}g}W_g \xrightarrow{(-1)^{|V||a|}\sigma_{V,a}\otimes\mathrm{id}_W} V_{\omega^{|V|+|W|}g}aW_g \xrightarrow{(\mathrm{id}_{VW}\otimes\rho^a(g))\circ(\mathrm{id}_V\otimes\sigma_{W,a})} V_{\omega^{|V|+|W|}g}W_g a,$$

where σ denotes the switch map in vector spaces and, for readability, we have dropped the subscripts on V and W in writing down the map. The sign $(-1)^{|V||a|}$ comes from the symmetry in $\mathrm{Rep}(G,\omega)$. This composes to:

$$aV_{\omega^{|V|+|W|}g}W_g \xrightarrow{(-1)^{|V||a|}(\mathrm{id}_{V\otimes W}\otimes\rho^a(g))\circ\sigma_{a,VW}} V_{\omega^{|V|+|W|}g}W_g a,$$

Observe that:

$$(-1)^{(|V|)|a|}\rho^a(g) = \rho^a(\omega^{|V|}g),$$

so the half-braiding becomes (using naturality of the switch map):

$$aV_{\omega^{|V|+|W|}g}W_g \xrightarrow{\sigma_{a,VW}\circ(\mathrm{id}_{V\otimes W}\otimes\rho^a(\omega^{|V|}g))} V_{\omega^{|V|+|W|}g}W_g a.$$

Now, comparing this with the definition of the half-braiding in $\mathbf{Vect}_G[G]$ (see Equation (A.40)), this indicates that $V_{\omega^{|V|+|W|}g}W_g$ is, in $V\otimes_s W$, a summand of the fibre over $\omega^{|V|}g$. We have found:

$$(V\otimes_s W)_{\omega^{|V|}g} = V_{\omega^{|V|+|W|}g}W_g.$$

or, reindexing:

$$(V \otimes_s W)_g = V_{\omega^{|W|}g} W_{\omega^{|V|}g}.$$

This concludes the proof of Theorem 2.35.

Chapter 3

The Drinfeld Centre as 2-Fold Tensor Category

3.1 Introduction

By the famous Eckmann-Hilton argument, any two compatible (mutually homomorphic) monoid structures on a set are commutative and equal. Similarly, one can show that any two strongly compatible monoidal structures on a category are naturally isomorphic and braided. If one relaxes the compatibility to be lax, (so not given by a natural isomorphism, but rather just a monoidal transformation), the Eckmann-Hilton argument no longer holds. This allows for the existence of lax 2-fold monoidal categories, and, in the linear case, lax 2-fold tensor categories.

The goal for this Chapter is to show that the Drinfeld centre of a symmetric fusion category is a 2-fold tensor category for its convolution (usual) tensor product together with the symmetric tensor product defined in Chapter 2. It turns out that, in fact, these tensor products are also oplaxly compatible, and that the lax and oplax structures are one-sided inverses for each other. This leads us to defining the notion of strongly inclusive bilax 2-fold monoidal category (Definition 3.2). Additionally, the braiding and symmetry for the convolution and symmetric tensor product are compatible with the lax structure. All in all, we will show (Theorem 3.5) that the Drinfeld centre of a symmetric fusion category is a vertically symmetric braided strongly inclusive bilax 2-fold tensor category.

The outline of this Chapter is as follows. We will first define, in Section 3.2.1, the notion of lax 2-fold monoidal category. We spell out what it means for such a category to be braided (or symmetric), and capture the extra compatibilities that the structures on the Drinfeld centre of a symmetric fusion category exhibit in definitions. The rest of this Chapter, Section 3.3, is then devoted to proving the main Theorem 3.5 of this Chapter.

3.2 Preliminaries

In this section we set up the theory of lax 2-fold monoidal categories.

3.2.1 Lax 2-fold monoidal categories

The following definition is inspired by [BFAV03], but allows for the units of the monoidal structure to be different.

Definition 3.1. Let $\mathcal{C}$ be a category, equipped with two monoidal structures $\otimes_1$ and $\otimes_2$, with units $\mathbb{I}_1$ and $\mathbb{I}_2$, respectively. The associator and right and left unitor isomorphisms for monoidal structures will be denoted $\alpha_1, \rho_1, \lambda_1$ and $\alpha_2, \rho_2, \lambda_2$, respectively. Then $(\mathcal{C}, \otimes_1, \otimes_2)$ is called *lax 2-fold monoidal* if there exists a natural transformation η with components

$$\eta_{c,c',d,d'} \colon (c \otimes_1 c') \otimes_2 (d \otimes_1 d') \to (c \otimes_2 d) \otimes_1 (c' \otimes_2 d'),$$

and morphisms

$$\begin{aligned} u_0 &\colon \mathbb{I}_2 \to \mathbb{I}_1 \\ u_1 &\colon \mathbb{I}_1 \otimes_2 \mathbb{I}_1 \to \mathbb{I}_1 \\ u_2 &\colon \mathbb{I}_2 \to \mathbb{I}_2 \otimes_1 \mathbb{I}_2. \end{aligned}$$

We will refer to these morphisms as *compatibility morphisms*. These morphisms are such that the following diagrams commute for all $c, c', d, d' \in \mathcal{C}$.

(a)
$$\begin{tikzcd} \mathbb{I}_2 \otimes_1 \mathbb{I}_1 \arrow[r, "u_0 \otimes \mathrm{id}"] \arrow[d, "\rho_1"] & \mathbb{I}_1 \otimes_1 \mathbb{I}_1 \arrow[d, "\rho_1"] \\ \mathbb{I}_2 \arrow[r, "u_0"] & \mathbb{I}_1 \end{tikzcd} ,$$

together with the corresponding diagram for λ_1, and the corresponding diagrams for λ_2 and ρ_2,

(b)
$$\begin{tikzcd} \mathbb{I}_2 \otimes_2 (d \otimes_1 d') \arrow[r, "\lambda_2"] \arrow[d, "u_2 \otimes_2 \mathrm{id}"] & d \otimes_1 d' \\ (\mathbb{I}_2 \otimes_1 \mathbb{I}_2) \otimes_2 (d \otimes_1 d') \arrow[r, "\eta"] & (\mathbb{I}_2 \otimes_2 d) \otimes_1 (\mathbb{I}_2 \otimes_2 d') \arrow[u, "\lambda_2 \otimes_2 \lambda_2"'] \end{tikzcd},$$

where λ_2 is the left-unitor for $\otimes_2$. We similarly require the corresponding diagrams for the right-unitor to commute.

(c)
$$\begin{tikzcd} (\mathbb{I}_1 \otimes_1 c') \otimes_2 (\mathbb{I}_1 \otimes_1 d') \arrow[r, "\eta"] \arrow[d, "\lambda_1 \otimes_2 \lambda_1"] & (\mathbb{I}_1 \otimes_2 \mathbb{I}_1) \otimes_1 (c' \otimes_2 d') \arrow[d, "u_1 \otimes_2 \mathrm{id}"] \\ c' \otimes_2 d' & \mathbb{I}_1 \otimes_1 (c' \otimes_2 d') \arrow[l, "\lambda_1"] \end{tikzcd},$$

where λ_1 denotes the left-unitor for $\otimes_1$, and the corresponding diagram for the right-unitor is also required to commute.

(d)
$$\begin{tikzcd}
(\mathbb{I}_1 \otimes_2 \mathbb{I}_1) \otimes_2 \mathbb{I}_1 \arrow[rr, "\alpha_2"] \arrow[d, "u_1 \otimes_2 \mathrm{id}"] & & \mathbb{I}_1 \otimes_2 (\mathbb{I}_1 \otimes_2 \mathbb{I}_1) \arrow[d, "\mathrm{id} \otimes_2 u_1"] \\
\mathbb{I}_1 \otimes_2 \mathbb{I}_1 \arrow[r, "u_1"] & \mathbb{I}_1 & \mathbb{I}_1 \otimes_2 \mathbb{I}_1, \arrow[l, "u_1"']
\end{tikzcd}$$

and the corresponding diagram for u_2 and α_1,

(e)
$$\begin{tikzcd}
((c \otimes_1 d) \otimes_2 (c' \otimes_1 d')) \otimes_2 (c'' \otimes_1 d'') \arrow[r, "\alpha_2"] \arrow[d, "\eta \otimes_2 \mathrm{id}"] & (c \otimes_1 d) \otimes_2 ((c' \otimes_1 d') \otimes_2 (c'' \otimes_1 d'')) \arrow[d, "\mathrm{id} \otimes_2 \eta"] \\
((c \otimes_2 c') \otimes_1 (d \otimes_2 d')) \otimes_2 (c'' \otimes_1 d'') \arrow[d, "\eta"] & (c \otimes_1 d) \otimes_2 ((c' \otimes_2 c'') \otimes_1 (d' \otimes_2 d'')) \arrow[d, "\eta"] \\
((c \otimes_2 c') \otimes_2 c'') \otimes_1 ((d \otimes_2 d') \otimes_2 d'') \arrow[r, "\alpha_2 \otimes_1 \alpha_2"] & (c \otimes_2 (c' \otimes_2 c'')) \otimes_1 (d \otimes_2 (d' \otimes_2 d'')),
\end{tikzcd}$$

where α_2 denotes the associator for $\otimes_2$.

(f)
$$\begin{tikzcd}
((c \otimes_1 c') \otimes_1 c'') \otimes_2 ((d \otimes_1 d') \otimes_1 d'') \arrow[r, "\alpha_1 \otimes_2 \alpha_1"] \arrow[d, "\eta"] & (c \otimes_1 (c' \otimes_1 c'')) \otimes_2 (d \otimes_1 (d' \otimes_1 d'')) \arrow[d, "\eta"] \\
((c \otimes_1 c') \otimes_2 (d \otimes_1 d')) \otimes_1 (c'' \otimes_2 d'') \arrow[d, "\eta"] & (c \otimes_2 d) \otimes_1 ((c' \otimes_1 c'') \otimes_2 (d' \otimes_1 d'')) \arrow[d, "\eta"] \\
((c \otimes_2 d) \otimes_1 (c' \otimes_2 d')) \otimes_1 (c'' \otimes_2 d'') \arrow[r, "\alpha_1"] & (c \otimes_2 d) \otimes_1 ((c' \otimes_2 d') \otimes_1 (c'' \otimes_2 d'')),
\end{tikzcd}$$

where α_1 denotes the associator for $\otimes_1$.

Definition 3.2. Dualising Definition 3.1 yields the notion of *oplax 2-fold monoidal*[1]. A category that is both lax and oplax 2-fold monoidal will be called *bilax 2-fold monoidal*.

If $\mathcal{C}$ is a bilax 2-fold monoidal category with lax compatibility morphisms η, u_0, u_1, u_2 and ζ, v_0, v_1, v_2 that satisfy

$$\eta \circ \zeta = \mathrm{id}, \quad u_0 \circ v_0 = \mathrm{id}, \quad u_1 \circ v_2 = \mathrm{id}, \quad v_1 \circ u_2 = \mathrm{id},$$

then $\mathcal{C}$ will be called an *inclusive bilax 2-fold monoidal category*. If, additionally, u_1 and v_2 are isomorphisms, we will call $\mathcal{C}$ *strongly inclusive*.

The notion of bilax 2-fold monoidal is known in the community. The notion of (strong) inclusivity is introduced here to capture the structure $\mathcal{Z}(\mathcal{A})$ has.

Definition 3.3. Let $(\mathcal{C}, \otimes_1, \otimes_2)$ be lax 2-fold monoidal and let β_1 be a braiding for $\otimes_1$. Then $\mathcal{C}$ is called *horizontally braided lax 2-fold monoidal* if the braiding is such that the following diagrams commute:

(a)
$$\begin{tikzcd}
\mathbb{I}_2 \arrow[r, "u_2"] \arrow[d, "u_2"] & \mathbb{I}_2 \otimes_1 \mathbb{I}_2 \arrow[dl, "\beta_1"] \\
\mathbb{I}_2 \otimes_1 \mathbb{I}_2 &
\end{tikzcd},$$

[1]This also corresponds to switching the roles of $\otimes_1$ and $\otimes_2$.

(b)
$$\begin{tikzcd}
(c \otimes_1 c') \otimes_2 (d \otimes_1 d') \arrow[r, "\eta"] \arrow[d, "\beta_1 \otimes_2 \beta_1"] & (c \otimes_2 d) \otimes_1 (c' \otimes_2 d') \arrow[d, "\beta_1"] \\
(c' \otimes_1 c) \otimes_2 (d' \otimes_1 d) \arrow[r, "\eta"] & (c' \otimes_2 d') \otimes_1 (c \otimes_2 d)
\end{tikzcd} .$$

If β_2 is a braiding for $\otimes_2$ and satisfies the analogous compatibility with η and u_1, we call $\mathcal{C}$ *vertically braided.* If $\mathcal{C}$ is both horizontally and vertically braided we will refer to it as *braided lax 2-fold monoidal.* If, in a braided lax 2-fold monoidal category, the horizontal (resp. vertical) braiding is symmetric the category will be called *horizontally (reps. vertically) symmetric.*

If $\mathcal{C}$ is additionally a bilax 2-fold monoidal category, and the horizontal or vertical braiding satisfies the corresponding compatibility with the oplax compatibility morphisms, $\mathcal{C}$ will be called *(horizontally or vertically) braided bilax 2-fold monoidal.*

Definition 3.4. When a lax 2-fold monoidal category $\mathcal{C}$ is enriched and tensored over **Vect**, and the monoidal structures are tensor structures, we will call $\mathcal{C}$ a *lax 2-fold tensor category.*

3.3 The Drinfeld Centre as a Lax 2-Fold Monoidal Category

This section is devoted to proving the main theorem of this Chapter:

Theorem 3.5. *Let $\mathcal{A}$ be a symmetric fusion category, and let $\otimes_c$ and $\otimes_s$ denote its usual and its symmetric tensor product (Definitions 2.11 and 2.22), respectively. Then $(\mathcal{Z}(\mathcal{A}), \otimes_c, \otimes_s)$ is a vertically symmetric braided strongly inclusive bilax 2-fold tensor category, cf. Definitions 3.1, 3.2, 3.3 and 3.4.*

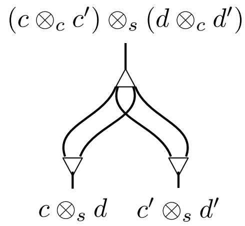
The structure of this section is as follows. We will first define the compatibility morphisms from Definition 3.1, we will denote these by (η, u_0, u_1, u_2) for the lax direction and (ζ, v_0, v_1, v_2) for the oplax direction. We will then proceed to check their coherence, combining the necessary proofs for the two cases whenever possible.

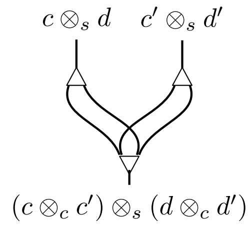
3.3.1 Lax Compatibility Morphisms

The comparison morphism

The following lemma allows us to define η and ζ.

Lemma 3.6. *Let $c, c', d, d' \in \mathcal{Z}(\mathcal{A})$, then the following string diagrams define* *morphisms in $\mathcal{Z}(\mathcal{A})$:*

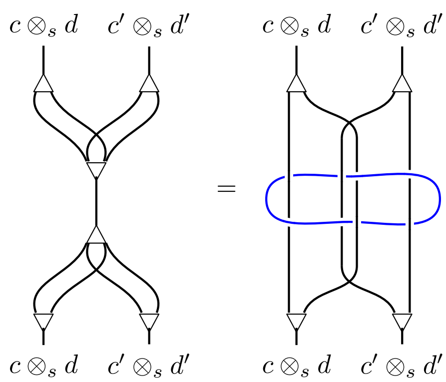
$\zeta_{c,d,c',d'} =$ 

and $\eta_{c,c',d,d'} =$ 

, (3.1)

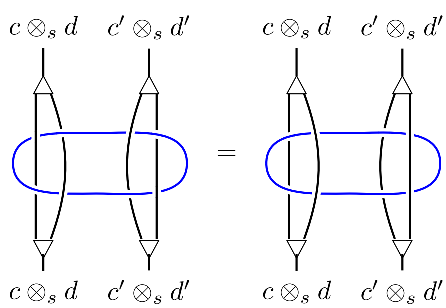
respectively. Here the unresolved crossing denotes the symmetry in $\mathcal{A}$, c.f. Section 2.2. These morphisms exhibit the object $(c \otimes_s d) \otimes_c (c' \otimes_s d')$ as a subobject of $(c \otimes_c c') \otimes_s (d \otimes_c d')$ with inclusion $\zeta_{c,d,c',d'}$ and projection $\eta_{c,c',d,d'}$.

Proof. We have to show that the composite along $(c \otimes_c c') \otimes_s (d \otimes_c d')$ of the two maps is the identity, and that they define morphisms in $\mathcal{Z}(\mathcal{A})$. For the former:



,

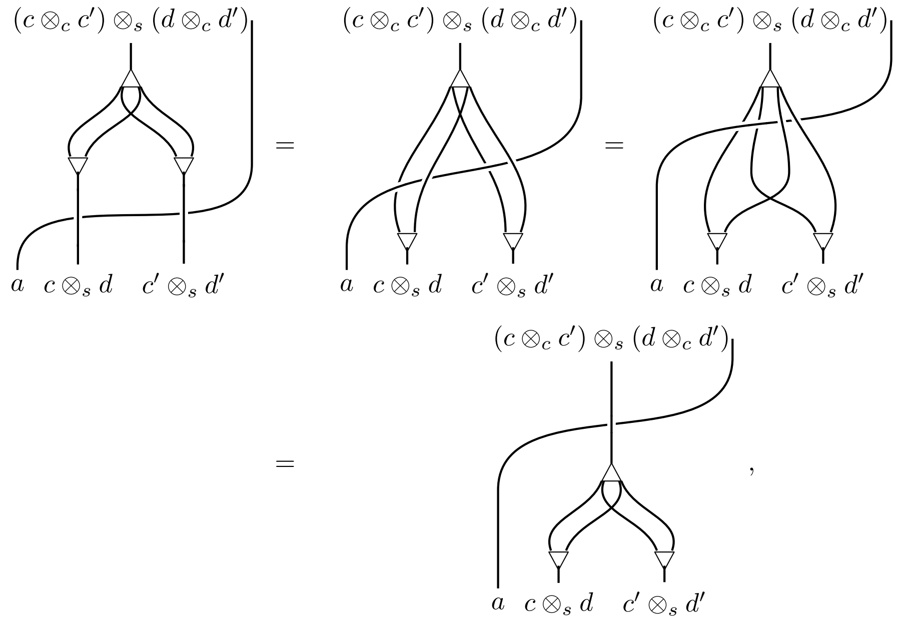
where we have replaced a projection followed by an inclusion with the idempotent from Lemma 2.9. We can now pull the top of the ring up and the bottom of the ring down, using the way the unresolved and resolved interact, see Equation (2.2), to get:



,

where in the last step we used the slicing from Lemma 2.12. In the last diagram, the ring comes out, and the diagram evaluates to the identity on $(c \otimes_s d) \otimes_c$

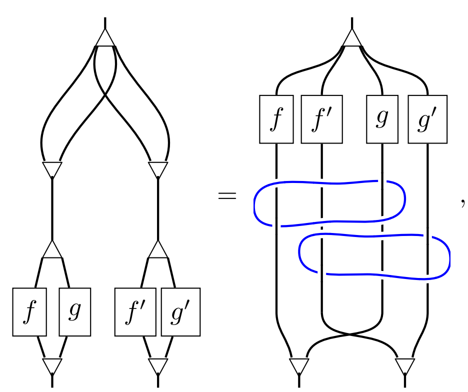
$(c' \otimes_s d')$, as desired. To show that the inclusion and projection are morphisms in $\mathcal{Z}(\mathcal{A})$, we compute, for $a \in \mathcal{A} \subset \mathcal{Z}(\mathcal{A})$:



where we made repeated use of slicing, and use of Equation (2.2) in the second equality. $\square$

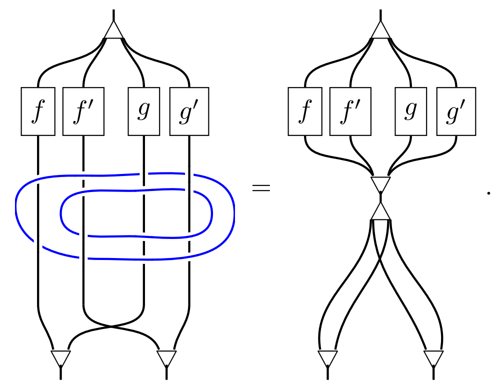
Lemma 3.7. *The morphisms from Equation* (3.1) *combine to give natural transformations.*

Proof. Let $f\colon c_1 \to c_2$, $f'\colon c'_1 \to c'_2$, $g\colon d_1 \to d_2$ and $g'\colon d'_1 \to d'_2$ be morphisms in $\mathcal{Z}(\mathcal{A})$. Using the definition of $\otimes_s$ on objects, Equation (2.22), we compute:



where we replaced projections followed by inclusions by the idempotent from Lemma 2.9, and used the naturality of the symmetry and braiding, as well as

Equation (2.2) to move the rings down and the morphisms up. We can now use cloaking (Lemma 2.8) for the bottom strand of the top ring with the bottom ring to get:



The last equality follows from the fact that the rings are transparent to each other. This we means we can bring the larger ring up, and the smaller ring down using Equation (2.2). We can then cancel them with the projection and inclusion respectively, cf. Equation (2.10).

The proof of naturality for the other map in Equation (3.1) is obtained by reading the diagrams top to bottom. □

With this Lemma in hand, we can define η and ζ to be the natural transformations with components $\eta_{c,c',d,d'}$ and $\zeta_{c,d,c',d'}$ defined in Equation (3.1).

Unit compatiblity

We will now produce the required morphisms u_0, u_1, u_2 and v_0, v_1, v_2 relating the units for the two tensor products on $\mathcal{Z}(\mathcal{A})$.

We start with the following observation:

Lemma 3.8. *The following are morphisms in* $\mathcal{Z}(\mathcal{A})$

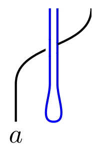
$$v_0\colon \quad \mathbb{I}_c \xrightarrow{\oplus \frac{t_i}{D} \mathrm{co\bar{e}v}_i} \bigoplus_{i \in \mathcal{O}(\mathcal{A})} i \otimes_c i^* =:$$ 

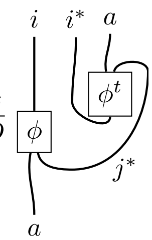
$$u_0\colon \quad \mathbb{I}_s \xrightarrow{\oplus \mathrm{ev}} \mathbb{I}_c =:$$  .

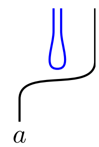
These morphism exhibit the unit $\mathbb{I}_c$ *for* $\otimes_c$ *as a subobject of the unit* $\mathbb{I}_s$ *for* $\otimes_s$ *with inclusion* v_0 *and projection* u_0*.*

Proof. The fact that these maps constitute a inclusion and projection pair is clear. We still need to show that these morphisms are morphisms in $\mathcal{Z}(\mathcal{A})$, i.e.

that they commute with the braiding. We compute:



$= \sum_{i,j\in\mathcal{O}(\mathcal{A})} \sum_{\phi\in B(aj,i)} \frac{t_j}{D}$


$=$


.

A similar, but simpler, argument shows that u_0 commutes with braiding. □

To produce the morphisms u_1 and v_2, we note that:

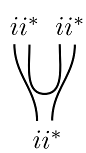
Lemma 3.9. *The objects $\mathbb{I}_c \otimes_s \mathbb{I}_c$ and $\mathbb{I}_c$ are canonically isomorphic.*

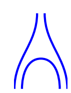
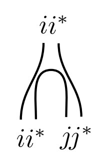
Proof. Recall that the object $\mathbb{I}_c \otimes_s \mathbb{I}_c$ is a subobject of $\mathbb{I}_c \otimes_c \mathbb{I}_c = \mathbb{I}_c$, and therefore canonically isomorphic to $\mathbb{I}_c$ (the convolution and symmetric product agree on $\mathcal{A} \subset \mathcal{Z}(\mathcal{A})$), as object in $\mathcal{A}$. We further observe that equipping $\mathbb{I}_c$ with the braiding (2.12) does not change its braiding. □

We set $u_1 : \mathbb{I}_c \otimes_s \mathbb{I}_c \overset{\cong}{\longleftrightarrow} \mathbb{I}_c : v_2$.
For u_2 and v_1, we use the following:

Lemma 3.10. *The object $\mathbb{I}_s$ is a subobject of $\mathbb{I}_s \otimes_c \mathbb{I}_s$, with inclusion and projection given by*

$u_2 =$

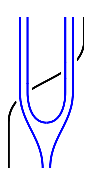
$= \sum_{i\in\mathcal{O}(\mathcal{A})} \frac{t_i}{D}$


and $v_1 =$

$= \sum_{i,j\in\mathcal{O}(\mathcal{A})} \delta_{i,j^*}$


,

repectively. δ_{i,j^*} denotes the Kronecker delta symbol that is 1 when $i = j^*$ and zero otherwise. In particular, u_2 and v_1 are morphisms in $\mathcal{Z}(\mathcal{A})$.

Proof. It is clear that u_2 and v_1 constitute an inclusion-projection pair, composing along of $\mathbb{I}_s \otimes_c \mathbb{I}_s$ gives $\sum_i d_i \frac{t_i}{D} = 1$ times the identity on $\mathbb{I}_s$. We still need to establish they are indeed morphisms in $\mathcal{Z}(\mathcal{A})$. That is, we need to show that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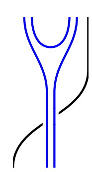
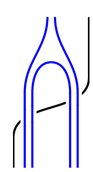
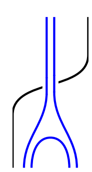
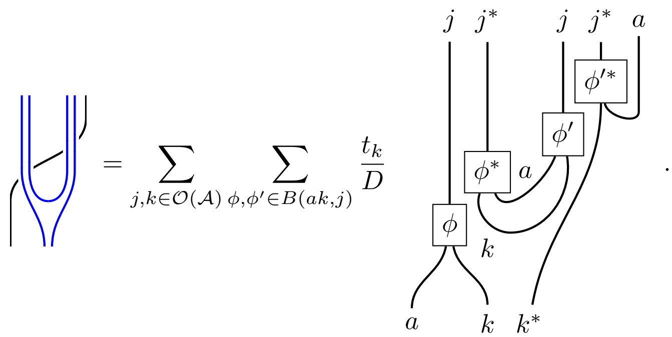
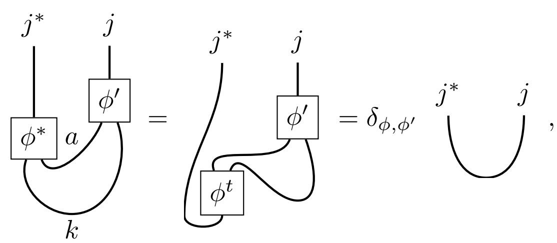

and

$=$

.

Unpacking the definition (Equation (2.16)) of the half-braiding for $\mathbb{I}_s$, we see that we get for $a \in \mathcal{A}$:

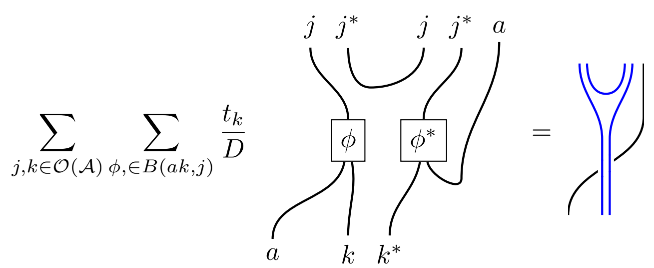


(3.2)

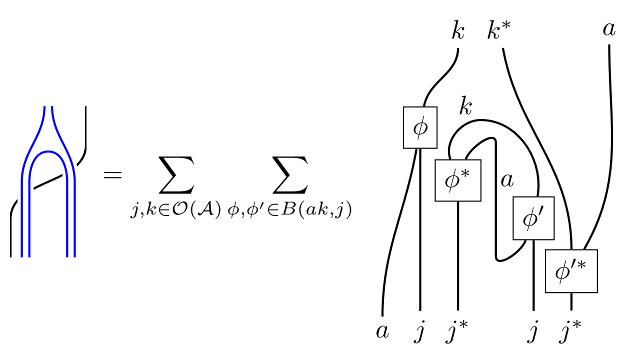
We can manipulate the middle part of the summands to see:



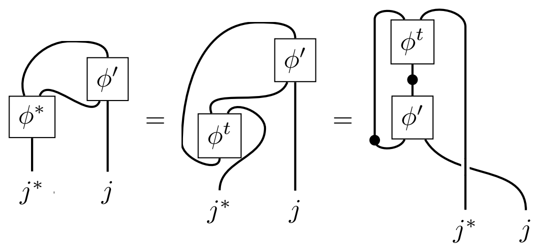
where $\phi, \phi' \in B(ak, j)$. Plugging this into Equation (3.2), we get:



Similarly, we have:



We can again examine the middle part of this diagram to see:



.

We can now use Equation 2.20 to move the twists to the j strand. Then, after composing with a coevaluation, we can view the last morphism as an endomorphism of j, so it is completely determined by its trace. This trace is $\delta_{\phi,\phi'}$, where we cancel the self-intersection with the twist. This means that this morphism evaluates to $\delta_{\phi,\phi'}$ times the evaluation on j^*j. Plugging this in yields the desired relation. □

3.3.2 Coherence

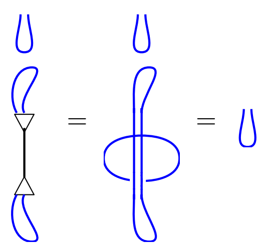
This section is devoted to proving that the morphisms from the previous section satisfy the coherence conditions from Definition 3.1. This will establish Theorem 3.5.

Unitor coherence

Lemma 3.11. *The morphisms u_0 and v_0 satisfy the coherence diagrams from Definition 3.1*(a)*, where $1 = c, 2 = s$ and $1 = s, 2 = c$ respectively.*

Proof. For u_0, both routes through the diagram in 3.1(a) evaluate to u_0 directly, so there is nothing to prove.

For v_0, it is more convenient to compare $\mathbb{I}_c \xrightarrow{\lambda_s^{-1}} \mathbb{I}_c \otimes_s \mathbb{I}_s \xrightarrow{u_0 \otimes_s \mathrm{id}} \mathbb{I}_s \otimes_s \mathbb{I}_s \xrightarrow{\lambda_s} \mathbb{I}_s$ to v_0. To do this, observe that, in string diagrams, this composite computes as:



,

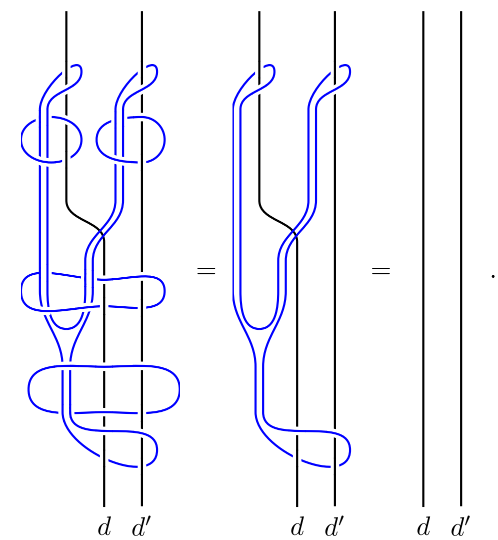
where we have applied snapping (Lemma 2.17). □

Lemma 3.12. *The morphisms η, u_2 make the diagrams from Definitions 3.1*(b) *commute for $\otimes_1 = \otimes_c$ and $\otimes_2 = \otimes_c$. Analogously, the morphisms ζ, v_1 make the diagrams from Definitions 3.1*(c) *commute for $\otimes_1 = \otimes_s$ and $\otimes_2 = \otimes_c$.*

Proof. Consider the anti-clockwise composite in the diagram from $d \otimes_c d'$:

$$d \otimes_c d' \xrightarrow{\lambda_s^{-1}} \mathbb{I}_s \otimes_s (d \otimes_c d') \xrightarrow{u_2 \otimes_s \mathrm{id}} (\mathbb{I}_s \otimes_c \mathbb{I}_s) \otimes_s (d \otimes_c d)$$
$$\xrightarrow{\eta_{\mathbb{I}_s, \mathbb{I}_s, d, d'}} (\mathbb{I}_s \otimes_s d) \otimes_c (\mathbb{I}_s \otimes_s d') \xrightarrow{\lambda_s \otimes_s \lambda_s} d \otimes_c d'.$$

In terms of string diagrams, replacing inclusions followed by projections by the idempotent from Lemma 2.9 right away, this becomes:



Here the first step is applying snapping (Lemma 2.17) and evaluating free loops to 1. The second equality is unwinding the loop, using that overcrossing for the loop is the symmetry in $\mathcal{A}$, hence the same as an unresolved crossing. Reading the diagrams top to bottom yields a proof for the case of ζ and v_1. □

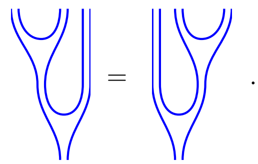
Associator coherence

Lemma 3.13. *The morphisms u_1 and u_2 satisfy the coherence diagrams from Definition 3.1*(d) *for $\otimes_1 = \otimes_c$ and $\otimes_2 = \otimes_s$. Furthermore, the morphisms v_1 and v_2 satisfy the coherence diagrams from Definition 3.1*(d) *for $\otimes_1 = \otimes_s$ and $\otimes_2 = \otimes_c$.*

Proof. For u_1 and v_1, there is nothing to prove. For u_2, we have to check that:

$$\begin{tikzcd}
(\mathbb{I}_1 \otimes_2 \mathbb{I}_1) \otimes_2 \mathbb{I}_1 \arrow[rr, "\alpha_2"] & & \mathbb{I}_1 \otimes_2 (\mathbb{I}_1 \otimes_2 \mathbb{I}_1) \\
\mathbb{I}_1 \otimes_2 \mathbb{I}_1 \arrow[u, "u_2 \otimes_2 \mathrm{id}"] \arrow[r, "u_1"] & \mathbb{I}_1 & \mathbb{I}_1 \otimes_2 \mathbb{I}_1 \arrow[u, "\mathrm{id} \otimes_2 u_2"'] \arrow[l, "u_1"]
\end{tikzcd}$$

commutes. In terms of string diagrams, this becomes:



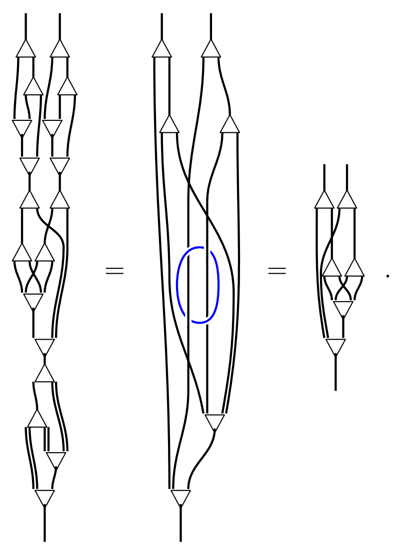
The proof for v_2 proceeds similarly, remembering that the associators for $\otimes_s$ are induced from the associators of $\mathcal{A}$. □

Lemma 3.14. *The morphisms η, u_1 make the diagrams from Definitions 3.1*(c) *commute for $\otimes_1 = \otimes_c$ and $\otimes_2 = \otimes_s$. Analogously, the morphisms ζ, v_2 make the diagrams from Definitions 3.1*(c) *commute for $\otimes_1 = \otimes_s$ and $\otimes_2 = \otimes_c$.*

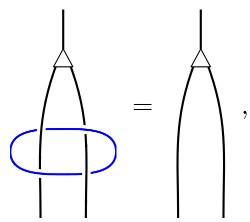
Proof. Using u_1 and v_1 are the isomorphisms between $\mathbb{I}_c$ and $\mathbb{I}_c \otimes_s \mathbb{I}_c$, we see there is nothing to prove. □

Lemma 3.15. *The natural transformation η makes the diagrams from Definitions 3.1*(e) *commute for $\otimes_1 = \otimes_c$ and $\otimes_2 = \otimes_s$. Analogously, the morphisms ζ make the diagrams from Definitions 3.1*(f) *commute for $\otimes_1 = \otimes_s$ and $\otimes_2 = \otimes_c$.*

Proof. For the first case, we compute the anti-clockwise composite from the top-right corner to the bottom-right corner:



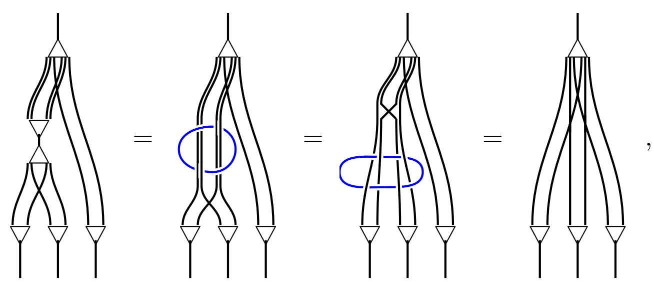
In the first step, we used the relation from Equation (2.10) to replace projection-inclusion pairs by rings, and subsequently used slicing (Lemma 2.12) to bring these rings to a position where we could use:



which is an easy consequence of Equation 2.10. This left the ring in the middle of the second diagram. To rid ourselves of this, we used the relation between the braiding in $\mathcal{Z}(\mathcal{A})$ and the symmetry in $\mathcal{A}$ from Equation (2.2) to cancel it with a projection. The third diagram is just the composite on the right hand side of the coherence diagram 3.1(e). Reading the diagrams in this proof top to bottom yields a proof of the commutativity of the diagram from Definition 3.1(f). □

Lemma 3.16. *The natural transformation η makes the diagrams from Definitions 3.1*(f) *commute for $\otimes_1 = \otimes_c$ and $\otimes_2 = \otimes_s$. Analogously, the morphisms ζ make the diagrams from Definitions 3.1*(e) *commute for $\otimes_1 = \otimes_s$ and $\otimes_2 = \otimes_c$.*

Proof. As we are suppressing the associators for $\otimes_c$ in the string diagrams, we see that we have, in terms of string diagrams:



for the left side composite in the diagram in Definition 3.1(f). Similar arguments also reduce the right side of this coherence diagram to the rightmost string diagram.

For the case involving ζ, we read the diagrams top to bottom. □

This finishes proving that $\mathcal{Z}(\mathcal{A})$ can be viewed as a bilax 2-fold monoidal category in as in Theorem 3.5.

Braiding coherence

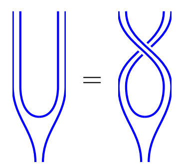
To prove Theorem 3.5, we still need to prove that the compatibility morphisms are compatible with the braiding.

Lemma 3.17. *The morphism v_2 makes the diagram from Definition 3.3*(a) *commute, where $\otimes_1 = \otimes_s$ and $\otimes_2 = \otimes_c$. Analogously, the morphism u_1 makes the corresponding diagram from Definition 3.3*(a) *for β_2 commute, where $\otimes_1 = \otimes_c$ and $\otimes_2 = \otimes_s$.*

Proof. As the maps involved are canonical isomorphisms coming from the unitors, the diagram 3.3(a) is automatically commutative. ☐

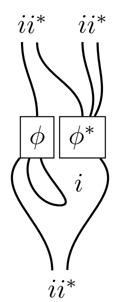
Lemma 3.18. *The morphism u_2 makes the diagram from Definition 3.3*(a) *commute, where $\otimes_1 = \otimes_c$ and $\otimes_2 = \otimes_s$. Analogously, the morphism v_1 makes the corresponding diagram from Definition 3.3*(a) *for β_2 commute, where $\otimes_1 = \otimes_s$ and $\otimes_2 = \otimes_c$.*

Proof. We need to show that:



Using the definition of the half-braiding on $\mathbb{I}_s$ from Equation (2.16), we see that the right hand side equals:

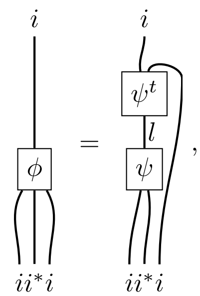
$$\sum_{i\in\mathcal{O}(\mathcal{A})}\ \sum_{\phi\in B(ii^*i,i)}$$



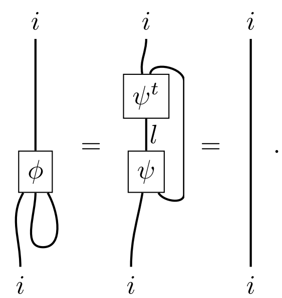
, (3.3)

where we have already taken into account restrictions on the possible combinations of labelling of the strands that can occur: the two leftmost strands coming into ϕ need to agree with the rightmost two coming out of ϕ^*, and these in turn must be labelled by a pair of dual objects. Further, as the leftmost strand going from bottom to top is a morphism between simple objects, its incoming and outgoing labels must be the same.

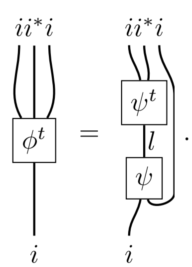
Examining $\phi \in \mathrm{Hom}(ii^*i, i) \cong \mathrm{Hom}(ii^*, ii^*)$, we see we can write it as:



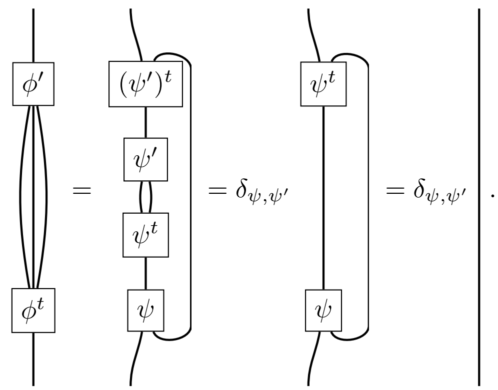
for some $l \in \mathcal{O}(\mathcal{A})$ and $\psi \in \mathrm{Hom}(ii^*, l)$. Therefore, picking a basis for $\mathrm{Hom}(ii^*, l)$ for each $l \in \mathcal{O}(\mathcal{A})$ gives a basis for $\mathrm{Hom}(ii^*, ii^*) \cong \mathrm{Hom}(ii^*i, i)$. Rescaling if necessary we can arrange



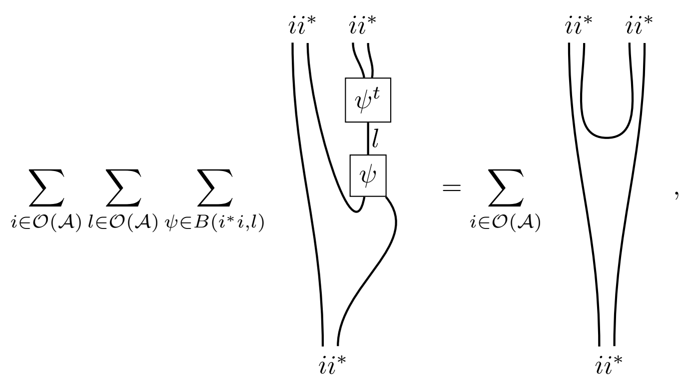
We now claim that the transposes for these ϕ are given by:



To see this, we compute the composite:



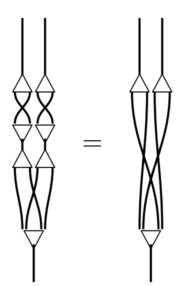
Putting this together, we see that the sum in Equation (3.3) becomes:



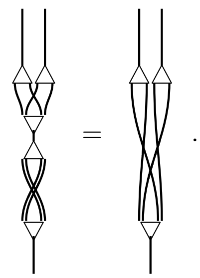
and this is what we wanted to show. For the proof of the other case, we read the diagrams top to bottom and see that two twists cancel. □

Lemma 3.19. *The morphism η makes the corresponding diagram from Definition 3.3*(b) *for β_2 commute, where $\otimes_1 = \otimes_c$ and $\otimes_2 = \otimes_s$. Analogously, the morphism ζ makes the diagram from Definition 3.3*(b) *commute, where $\otimes_1 = \otimes_s$ and $\otimes_2 = \otimes_c$.*

Proof. For the first statement, the top route in the diagram computes as:



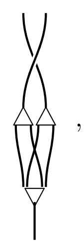
where we immediately cancelled the rings coming from the projection after inclusion (like in the proof of Lemma 3.15). Similarly, the bottom route computes as:



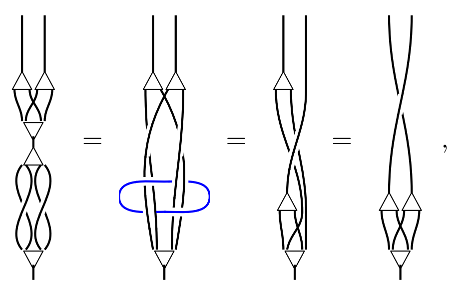
Reading the diagrams top to bottom yields a proof for the other assertion in the lemma. □

Lemma 3.20. *The morphism η makes the diagram from Definition 3.3*(b) *commute, where $\otimes_1 = \otimes_c$ and $\otimes_2 = \otimes_s$. Analogously, the morphism ζ makes the corresponding diagram from Definition 3.3*(b) *for β_2 commute, where $\otimes_1 = \otimes_s$ and $\otimes_2 = \otimes_c$.*

Proof. The top route computes as:



our goal is to show that the bottom route in the diagram is the same. For this composite we have that:



where in the first equality we slid the ring resulting from the projection-inclusion pair down and the second equality uses slicing (Lemma 2.12) to bring the ring out. For the analogous statement for ζ, we read the diagrams top to bottom. □

This completes the proof of the main Theorem 3.5.

Chapter 4

$\mathcal{Z}(\mathcal{A})$-Crossed Braided Categories

In this Chapter we will show how to obtain $\mathcal{Z}(\mathcal{A})$-crossed braided categories

4.1 Introduction

In the previous Chapter 3, we have shown that the Drinfeld centre $\mathcal{Z}(\mathcal{A})$ of a symmetric fusion category $\mathcal{A}$ is lax 2-fold monoidal for the convolution and symmetric tensor products. In this Chapter, we will examine what extra structure this 2-fold product gives to categories enriched over $(\mathcal{Z}(\mathcal{A}), \otimes_s)$. In particular, we will define, for such categories, a notion of monoidal structure that factors, on morphisms, through the convolution tensor product. We will refer to this as a $\mathcal{Z}(\mathcal{A})$-crossed monoidal structure (Definition 4.16). Additionally, we spell out what it means for such a monoidal structure to be braided (Definition 4.20).

We are, in part, motivated by the desire to, in Chapter 5, take a braided fusion category containing $\mathcal{A}$ and produce from this a braided object enriched over $\mathcal{A}$. It turns out that $\mathcal{Z}(\mathcal{A})$-crossed braided is the right notion to consider. On the other hand, in his book on Homotopy Quantum Field Theory [Tur10, Chapter VI], Turaev defined the notion of G-crossed braided fusion category (see Definition 4.29 here). We will show that, for $\mathcal{A}$ Tannakian, a $\mathcal{Z}(\mathcal{A})$-crossed braided category gives rise to such a G-crossed braided category. Furthermore, for $\mathcal{A}$ super-Tannakian, we define the notion of a super G-crossed braided category for a super-group (G, ω) (Definition 4.31), and show that, similarly, a $\mathcal{Z}(\mathcal{A})$-crossed braided category gives rise to a super G-crossed braided category. These two results constitute Proposition 4.36. We show that this construction has an inverse, this is the main Theorem 4.27 of this Chapter.

The structure of this Chapter is as follows. We will first, in Section 4.2, develop the theory of $\mathcal{Z}(\mathcal{A})$-crossed braided categories. In Section 4.3, is devoted to stating the definition of a (super) G-crossed braided category, and proving Theorem 4.36, which tells us $\mathcal{Z}(\mathcal{A})$-crossed braided categories give rise to (super)-G-crossed braided categories. After this, in Section 4.3.3, we construct an inverse to this construction, and prove in Section 4.3.4 our main Theorem 4.27.

4.2 $\mathcal{Z}(\mathcal{A})$-crossed braided categories

We now set up the theory of $\mathcal{Z}(\mathcal{A})$-crossed braided categories.

Notation 4.1. We will write $\mathcal{Z}(\mathcal{A})_s$ for the Drinfeld centre of a symmetric ribbon fusion category $\mathcal{A}$ equipped with the symmetric tensor product from Theorem 2.22.

4.2.1 $\mathcal{Z}(\mathcal{A})_s$-enriched categories

As $\mathcal{Z}(\mathcal{A})$ is abelian, categories enriched and tensored over $\mathcal{Z}(\mathcal{A})_s$ admit a notion of semi-simplicity. We will will assume that all our $\mathcal{Z}(\mathcal{A})_s$-enriched and tensored categories are semi-simple with finitely many simples.

Definition 4.2. Let $\mathcal{K}$ be a $\mathcal{Z}(\mathcal{A})_s$-enriched category. Then the *associated $\mathcal{A}$-enriched category* $\overline{\mathcal{K}}$ for $\mathcal{K}$ is the category obtained by applying the forgetful functor $\mathcal{Z}(\mathcal{A})_s \to \mathcal{A}$.

Lemma 4.3. *$\overline{\mathcal{K}}$ is indeed an $\mathcal{A}$-enriched category.*

Proof. By Proposition 2.23, the forgetful functor is lax monoidal, so this is a direct consequence of Proposition A.23. □

The assignment $\mathcal{K} \mapsto \overline{\mathcal{K}}$ interacts in the following way with the enriched Cartesian product:

Lemma 4.4. *For $\mathcal{K}$ and $\mathcal{L}$ be $\mathcal{Z}(\mathcal{A})_s$-enriched categories, we have that the map*

$$\overline{H} \colon \overline{\mathcal{K} \underset{s}{\boxtimes} \mathcal{L}} \to \overline{\mathcal{K}} \underset{\mathcal{A}}{\boxtimes} \overline{\mathcal{L}},$$

which acts the identity on objects and as the image under the forgetful functor of η (see Equation (3.1)) on hom-objects, is an $\mathcal{A}$-enriched functor. It has a left-sided inverse:

$$\overline{Z} \colon \overline{\mathcal{K}} \underset{\mathcal{A}}{\boxtimes} \overline{\mathcal{L}} \to \overline{\mathcal{K} \underset{s}{\boxtimes} \mathcal{L}},$$

which is also the identity on objects, and acts as ζ (Equation (3.1)) on hom-objects.

Proof. Observing that the image of η under the forgetful functor gives the lax monoidality morphism for the forgetful functor, the first part is a direct consequence of Proposition A.26. That $\overline{Z}$ is a left-sided inverse follows from $\zeta\eta = \mathrm{id}$ and Proposition A.24. □

$\mathcal{A}$-tensoring

We will need the following fact about the interaction between the other tensor product $\otimes_c$ on $\mathcal{Z}(\mathcal{A})$ and the $\mathcal{Z}(\mathcal{A})_s$-enriched and tensored structure.

Proposition 4.5. *Let $\mathcal{K}$ be an $\mathcal{Z}(\mathcal{A})_s$-enriched and tensored category, and denote its $\mathcal{Z}(\mathcal{A})_s$-tensoring by $\cdot$. Furthermore, we have for $a \in \mathcal{A} \subset \mathcal{Z}(\mathcal{A})$:*

$$a \otimes_c \mathcal{K}(-,k) \overset{\cong}{\Rightarrow} \mathcal{K}(-,(a \otimes_c \mathbb{I}_s) \cdot k). \tag{4.1}$$

Proof. By Lemma 2.27, for $z \in \mathcal{Z}(\mathcal{A})$ and $a \in \mathcal{A}$, we have:

$$a \otimes_c z \cong (a \otimes_c \mathbb{I}_s) \otimes_s z.$$

By Lemma A.13, this means that we have for all $k, k' \in \mathcal{K}$:

$$a \otimes_c \mathcal{K}(k,k') \cong \mathcal{K}(k,(a \otimes_c \mathbb{I}_s) \cdot k').$$

□

Definition 4.6. The *2-category $\mathcal{Z}(\mathcal{A})$**LinCat** of $\mathcal{Z}(\mathcal{A})_s$-enriched and tensored and $\mathcal{A}$-tensored categories* has morphisms $\mathcal{Z}(\mathcal{A})$-enriched functors $F\colon \mathcal{K} \to \mathcal{K}'$ which respect the $\mathcal{Z}(\mathcal{A})$-tensoring, and 2-morphisms $\mathcal{Z}(\mathcal{A})$-enriched natural tranformations η satisfying $\eta_{ak} = \mathrm{id}_a \eta_k$.

This 2-category admits a 2-functor:

$$\overline{(-)}\colon \mathcal{Z}(\mathcal{A})\mathbf{LinCat} \to \mathcal{A}\mathbf{LinCat},$$

where $\mathcal{A}$**LinCat** was defined in Definition A.6.

As this is a 2-category of categories enriched over a symmetric category, it comes equipped with a symmetric monoidal structure, see Definition A.17.

Definition 4.7. We will denote by $\underset{s}{\boxtimes}$ the *Cauchy completion of the enriched Cartesian product of $\mathcal{Z}(\mathcal{A})_s$-enriched and tensored categories.* (The notion of Cauchy completion is defined in Definition A.15.)

4.2.2 Crossed Product of $\mathcal{Z}(\mathcal{A})_s$-Enriched Categories

Definition 4.8. Let $\mathcal{K}, \mathcal{L}$ be categories enriched over $\mathcal{Z}(\mathcal{A})_s$. Then the *convolution product $\mathcal{K} \underset{c}{\boxtimes} \mathcal{L}$ of $\mathcal{K}$ and $\mathcal{L}$* is the Cauchy completion of the $(\mathcal{Z}(\mathcal{A}), \underset{s}{\otimes})$-enriched category with objects symbols $k \boxtimes l$ for $k \in \mathcal{K}$ and $l \in \mathcal{L}$, and hom-objects

$$\mathcal{K} \underset{c}{\boxtimes} \mathcal{L}(k \boxtimes l, k' \boxtimes l') := \mathcal{K}(k,k') \underset{c}{\otimes} \mathcal{L}(l,l').$$

The composition is defined by the composite of the projection η from Equation (3.1) and the compositions in $\mathcal{K}$ and $\mathcal{L}$. The identity morphisms are given by the composite

$$\mathrm{I}_{k \boxtimes l}\colon \mathbb{I}_s \xrightarrow{u_2} \mathbb{I}_s \underset{c}{\otimes} \mathbb{I}_s \xrightarrow{\mathrm{I}_k \underset{c}{\otimes} \mathrm{I}_l} \mathcal{K}(k,k) \underset{c}{\otimes} \mathcal{L}(l,l), \tag{4.2}$$

where I_k and I_l correspond to the identity morphisms on k and l, respectively. The morphism u_2 is as in Lemma 3.10.

This is equivalenty the Cauchy completion of the category obtained by change of basis along the functor $\otimes_c \colon \mathcal{Z}(\mathcal{A})_s \boxtimes \mathcal{Z}(\mathcal{A})_s \to \mathcal{Z}(\mathcal{A})_s$ for the Deligne tensor product $\mathcal{K} \boxtimes \mathcal{L}$, and hence it is $\mathcal{Z}(\mathcal{A})_s$-enriched by Proposition A.23. Similarly to the situation for linear categories, the Cauchy completion ensures that the resulting category is semi-simple with finitely many simples. We will provide a separate proof for convenience of the reader. Also, it contains an awesome diagram.

Lemma 4.9. *The composition in $\mathcal{K} \underset{c}{\boxtimes} \mathcal{L}$ is associative, and the identity morphisms are given by Equation* (4.2)*.*

Proof. For readability of this proof, we will use the shorthand $\mathcal{K}_{ij} := \mathcal{K}(k_i, k_j)$ with $i, j = 0, 1, 2, 3$, and the obvious version of this for $\mathcal{L}$. We will further suppress $\otimes_s$ from the notation for this proof and write $\cdot$ for $\underset{c}{\otimes}$.

For associativity, we should check that the outside of the following diagram commutes:

$$\begin{tikzcd}
((\mathcal{K}_{23} \cdot \mathcal{L}_{23})(\mathcal{K}_{12} \cdot \mathcal{L}_{12}))(\mathcal{K}_{01} \cdot \mathcal{L}_{01}) \arrow[rr, "\alpha"] \arrow[d, "\eta"] & & (\mathcal{K}_{23} \cdot \mathcal{L}_{23})((\mathcal{K}_{12} \cdot \mathcal{L}_{12})(\mathcal{K}_{01} \cdot \mathcal{L}_{01})) \arrow[d, "\eta"] \\
((\mathcal{K}_{23}\mathcal{K}_{12}) \cdot (\mathcal{L}_{23}\mathcal{L}_{12}))(\mathcal{K}_{01} \cdot \mathcal{L}_{01}) \arrow[r, "\eta"] \arrow[d, "\circ"] & ((\mathcal{K}_{23}\mathcal{K}_{12})\mathcal{K}_{01}) \cdot ((\mathcal{L}_{23}\mathcal{L}_{12})\mathcal{L}_{01}) \arrow[d, "\alpha"] \arrow[ddl, "\circ"] & (\mathcal{K}_{23} \cdot \mathcal{L}_{23})((\mathcal{K}_{12}\mathcal{K}_{01}) \cdot (\mathcal{L}_{12}\mathcal{L}_{01})) \arrow[dl, "\eta"] \arrow[d, "\circ"] \\
(\mathcal{K}_{13} \cdot \mathcal{L}_{13})(\mathcal{K}_{01} \cdot \mathcal{L}_{01}) \arrow[d, "\eta"] & (\mathcal{K}_{23}(\mathcal{K}_{12}\mathcal{K}_{01})) \cdot (\mathcal{L}_{23}(\mathcal{L}_{12}\mathcal{L}_{01})) \arrow[dr, "\circ"] & (\mathcal{K}_{23}(\mathcal{K}_{12}\mathcal{K}_{01})) \cdot (\mathcal{L}_{23}(\mathcal{L}_{12}\mathcal{L}_{01})) \arrow[d, "\eta"] \\
(\mathcal{K}_{13}\mathcal{K}_{01}) \cdot (\mathcal{L}_{13}\mathcal{L}_{01}) \arrow[r, "\circ"] & \mathcal{K}_{03} \cdot \mathcal{L}_{03} & (\mathcal{K}_{23}(\mathcal{K}_{12}\mathcal{K}_{01})) \cdot (\mathcal{L}_{23}(\mathcal{L}_{12}\mathcal{L}_{01})). \arrow[l, "\circ"]
\end{tikzcd}$$

The top right face of this diagram is part (e) from Definition 3.1, so commutes by Theorem 3.5. The left and right faces in middle commute by naturality of η, whereas the bottom left face commutes by associativity of $\circ$ in $\mathcal{K}$ and $\mathcal{L}$. This establishes that the composition in $\mathcal{K} \underset{c}{\boxtimes} \mathcal{L}$ is associative in the appropriate sense.

To establish the morphisms from (4.2) indeed define the identity morphisms, we need to check that the composite

$$\begin{aligned}
&\mathcal{K} \underset{c}{\boxtimes} \mathcal{L}(k_0 \boxtimes l_0, k_1 \boxtimes l_1) \xrightarrow{\rho_s} \mathcal{K} \underset{c}{\boxtimes} \mathcal{L}(k_0 \boxtimes l_0, k_1 \boxtimes l_1)\mathbb{I}_s \xrightarrow{\mathrm{I}_{k_0 \boxtimes l_0}} \\
&\qquad \mathcal{K} \underset{c}{\boxtimes} \mathcal{L}(k_0 \boxtimes l_0, k_1 \boxtimes l_1)\mathcal{K} \underset{c}{\boxtimes} \mathcal{L}(k_0 \boxtimes l_0, k_0 \boxtimes l_0) \xrightarrow{\circ} \mathcal{K} \underset{c}{\boxtimes} \mathcal{L}(k_0 \boxtimes l_0, k_1 \boxtimes l_1)
\end{aligned}$$

is the identity. Unpacking the definitions of the maps involved, we see that we need to check that the outside of the following diagram commutes:

$$\begin{tikzcd}
\mathcal{K}_{01} \cdot \mathcal{L}_{01} \arrow[d, "\rho_s^{-1}"] \arrow[drr, "\rho_s^{-1} \cdot \rho_s^{-1}"] & & & & (\mathcal{K}_{01}\mathcal{K}_{00}) \cdot (\mathcal{L}_{01}\mathcal{L}_{00}) \arrow[llll, "\circ"'] \\
 & & (\mathcal{K}_{01}\mathbb{I}_s) \cdot (\mathcal{L}_{01}\mathbb{I}_s) \arrow[urr, "\mathrm{I}_{k_0 \boxtimes l_0}"] & & \\
(\mathcal{K}_{01} \cdot \mathcal{L}_{01})\mathbb{I}_s \arrow[rr, "u_2"] & & (\mathcal{K}_{01} \cdot \mathcal{L}_{01})(\mathbb{I}_s \cdot \mathbb{I}_s) \arrow[u, "\eta"] \arrow[rr, "\mathrm{I}_{k_0} \cdot \mathrm{I}_{k_0}"] & & (\mathcal{K}_{01} \cdot \mathcal{L}_{01})(\mathcal{K}_{00} \cdot \mathcal{L}_{00}). \arrow[uu, "\eta"']
\end{tikzcd}$$

The right face of this commutes by naturality of η, the top face because $\circ$ and I are compatible in $\mathcal{K}$ and $\mathcal{L}$. The left face is (d) from Definition 3.1. $\square$

We remind the reader that the forgetful functor $\mathcal{Z}(\mathcal{A}) \to \mathcal{A}$ takes $\otimes_c$ to $\otimes_{\mathcal{A}}$.

Lemma 4.10. *If $\mathcal{K}$ and $\mathcal{L}$ are $\mathcal{Z}(\mathcal{A})_s$-enriched and tensored, then* $\overline{\mathcal{K} \underset{c}{\boxtimes} \mathcal{L}} = \overline{\mathcal{K}} \underset{\mathcal{A}}{\boxtimes} \overline{\mathcal{L}}$.

Proof. To prove the claim, observe that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{Z}(\mathcal{A}) \boxtimes \mathcal{Z}(\mathcal{A}) \arrow[r, "\otimes_c"] \arrow[d, "\mathbf{Forget} \boxtimes \mathbf{Forget}"'] & \mathcal{Z}(\mathcal{A}) \arrow[d, "\mathbf{Forget}"] \\
\mathcal{A} \arrow[r, "\otimes_{\mathcal{A}}"] & \mathcal{A},
\end{tikzcd}$$

because Forget is a strictly monoidal functor with respect to $\otimes_c$. This means $\overline{\mathcal{K} \underset{c}{\boxtimes} \mathcal{L}}$ and $\overline{\mathcal{K}} \underset{\mathcal{A}}{\boxtimes} \overline{\mathcal{L}}$ are the result of a change of basis along equal functors, and hence the same category. $\square$

We also have:

Corollary 4.11. *If $\mathcal{K}$ and $\mathcal{L}$ are $\mathcal{Z}(\mathcal{A})_s$-enriched and tensored, then $\mathcal{K} \underset{c}{\boxtimes} \mathcal{L}$ is $\mathcal{Z}(\mathcal{A})_s$-tensored, with tensoring*

$$a(k \boxtimes l) := (ak \boxtimes l) \cong (k \boxtimes al).$$

Proof. This follows from Proposition A.18. $\square$

As long as we restrict our attention to categories that are $\mathcal{Z}(\mathcal{A})_s$-tensored, the unit for the convolution product is $\mathcal{A}$ enriched over $\mathcal{A} \subset \mathcal{Z}(\mathcal{A})_s$, denoted by $\mathcal{A}_{\mathcal{Z}}$. If we drop the tensoring, the unit would become the one object category with endomorphism object for this single object $\mathbb{I}_c \in \mathcal{Z}(\mathcal{A})$. This category is not $\mathcal{Z}(\mathcal{A})_s$-tensored, and taking the free $\mathcal{Z}(\mathcal{A})$-enriched and $\mathcal{Z}(\mathcal{A})_s$-tensored category on this gives $\mathcal{A}_{\mathcal{Z}}$.

Lemma 4.12. *The convolution product of $\mathcal{A}_{\mathcal{Z}}$ with any $\mathcal{Z}(\mathcal{A})_s$-enriched and tensored category $\mathcal{K}$ is equivalent to $\mathcal{K}$.*

Proof. From Proposition 4.5, we get a functor

$$\begin{aligned} \mathcal{A}_{\mathcal{Z}} \underset{c}{\boxtimes} \mathcal{K} &\to \mathcal{K} \\ a \boxtimes k &\mapsto (a \otimes_c \mathbb{I}_s) \cdot k, \end{aligned}$$

which on morphisms is the composite (using Equation 4.1, Lemmas 2.27, A.13 and A.9, and the adjunction for duals in $\mathcal{A}$)

$$\begin{aligned} \mathcal{A}_{\mathcal{Z}}(a, a') \otimes_c \mathcal{K}(k, k') &\cong a^* \otimes_c \mathcal{A}_{\mathcal{Z}}(\mathbb{I}_{\mathcal{A}}, a') \otimes_c \mathcal{K}(k, k') \\ &\cong \mathcal{K}(k, (a^* \otimes_c a' \otimes_c \mathbb{I}_s) \cdot k') \\ &\cong \mathcal{K}(k, (a^* \otimes_c \mathbb{I}_s) \cdot (a' \otimes_c \mathbb{I}_s) \cdot k') \\ &\cong \mathcal{K}((a \otimes_c \mathbb{I}_s) \cdot k, (a' \otimes_c \mathbb{I}_s) \cdot k'). \end{aligned}$$

This is fully faithful (induces isomorphisms on hom-objects) by construction, and seen to be essentially surjective by taking $a = \mathbb{I}_c$, hence an equivalence. □

Definition 4.13. Let $\mathcal{K}, \mathcal{L}$ be categories enriched over $\mathcal{Z}(\mathcal{A})_s$. Then the braiding functor

$$B \colon \mathcal{K} \underset{c}{\boxtimes} \mathcal{L} \to \mathcal{L} \underset{c}{\boxtimes} \mathcal{K}$$

is given by $k \boxtimes l \mapsto l \boxtimes k$ on objects and by the braiding in $\mathcal{Z}(\mathcal{A})$ on Hom-objects.

Lemma 4.14. *B indeed defines a $\mathcal{Z}(\mathcal{A})_s$-enriched functor. Furthermore, this functor is an equivalence.*

Proof. Viewing $\mathcal{K} \underset{c}{\boxtimes} \mathcal{L}$ as coming from a change of basis (Proposition A.24) on $\mathcal{K} \boxtimes \mathcal{L}$ along $\otimes_c$ from $\mathcal{Z}(\mathcal{A})_s \boxtimes \mathcal{Z}(\mathcal{A})_s$ to $\mathcal{Z}(\mathcal{A})_s$, we notice we can get $\mathcal{L} \underset{c}{\boxtimes} \mathcal{K}$ from a change of basis on $\mathcal{K} \boxtimes \mathcal{L}$ along $\otimes_c$ precomposed with the **LinCat**-switch functor in $\mathcal{Z}(\mathcal{A})_s \boxtimes \mathcal{Z}(\mathcal{A})_s \to \mathcal{Z}(\mathcal{A})_s \boxtimes \mathcal{Z}(\mathcal{A})_s$. By definition, the braiding in $\mathcal{Z}(\mathcal{A})_c$ gives a natural isomorphism between these two functors. Hence, by Proposition A.30, if the braiding is lax monoidal with respect to $\otimes_s$, the braiding will induce the functor B, and this will be an equivalence. But the lax monoidality of the braiding is exactly what Definition 3.3 entails, so by Theorem 3.5 we are done. □

The lax and oplax compatibility morphisms for the 2-fold monoidal structure (Equation 3.1) on $\mathcal{Z}(\mathcal{A})$ give functors relating the convolution product and the enriched cartesian product of $\mathcal{Z}(\mathcal{A})_s$-enriched categories.

Proposition 4.15. *The assignments*

$$\begin{gathered} Z \colon (\mathcal{K} \underset{s}{\boxtimes} \mathcal{L}) \underset{c}{\boxtimes} (\mathcal{K}' \underset{s}{\boxtimes} \mathcal{L}') \leftrightarrow (\mathcal{K} \underset{c}{\boxtimes} \mathcal{K}') \underset{s}{\boxtimes} (\mathcal{L} \underset{c}{\boxtimes} \mathcal{L}') : H \\ k \boxtimes l \boxtimes k' \boxtimes l' \leftrightarrow k \boxtimes k' \boxtimes l \boxtimes l' \\ (\mathcal{K}_{01}\mathcal{L}_{01}) \cdot (\mathcal{K}'_{01}\mathcal{L}'_{01}) \underset{\eta}{\overset{\zeta}{\leftrightarrows}} (\mathcal{K}_{01} \cdot \mathcal{K}'_{01})(\mathcal{L}_{01} \cdot \mathcal{L}'_{01}), \end{gathered}$$

where we have used the notation from the proof of Lemma 4.9, are $\mathcal{Z}(\mathcal{A})_s$-enriched functors.

Proof. Composition in the category on the left hand side is given by:

$$\begin{aligned}((\mathcal{K}_{12}\mathcal{L}_{12})\cdot(\mathcal{K}'_{12}\mathcal{L}'_{12}))((\mathcal{K}_{01}\mathcal{L}_{01})\cdot(\mathcal{K}'_{01}\mathcal{L}'_{01})) &\xrightarrow{\eta} (\mathcal{K}_{12}\mathcal{L}_{12}\mathcal{K}_{01}\mathcal{L}_{01})\cdot(\mathcal{K}'_{12}\mathcal{L}'_{12}\mathcal{K}'_{01}\mathcal{L}'_{01}) \\ &\xrightarrow{s\cdot s}(\mathcal{K}_{12}\mathcal{K}_{01}\mathcal{L}_{12}\mathcal{L}_{01})\cdot(\mathcal{K}'_{12}\mathcal{K}'_{01}\mathcal{L}'_{12}\mathcal{L}'_{01}) \xrightarrow{(\circ\circ)\cdot(\circ\circ)} (\mathcal{K}_{02}\mathcal{L}_{02})\cdot(\mathcal{K}'_{02}\mathcal{L}'_{02}).\end{aligned}$$

On the right hand side, it is the composite:

$$\begin{aligned}&(\mathcal{K}_{12}\cdot\mathcal{K}'_{12})(\mathcal{L}_{12}\cdot\mathcal{L}'_{12})(\mathcal{K}_{01}\cdot\mathcal{K}'_{01})(\mathcal{L}_{01}\cdot\mathcal{L}'_{01}) \\ &\xrightarrow{s}(\mathcal{K}_{12}\cdot\mathcal{K}'_{12})(\mathcal{K}_{01}\cdot\mathcal{K}'_{01})(\mathcal{L}_{12}\cdot\mathcal{L}'_{12})(\mathcal{L}_{01}\cdot\mathcal{L}'_{01}) \\ &\xrightarrow{\eta\eta}((\mathcal{K}_{12}\mathcal{K}_{01})\cdot(\mathcal{K}'_{12}\mathcal{K}'_{01}))((\mathcal{L}_{12}\mathcal{L}_{01})\cdot(\mathcal{L}'_{12}\mathcal{L}'_{01})) \\ &\xrightarrow{(\circ\cdot\circ)(\circ\cdot\circ)}(\mathcal{K}_{02}\cdot\mathcal{K}'_{02})(\mathcal{L}_{02}\cdot\mathcal{L}'_{02}).\end{aligned}$$

Now, $\eta \otimes_s \eta$ gives a morphism from the first term in the second chain to the first term in the first chain, while η gives a morphism between the last terms. Functoriality of H is equivalent to the diagram formed in this way commuting. Comparing the penultimate terms in the sequences, we see that η gives a map between these, and the square this forms with the composition morphisms and the final η commutes by naturality of η. We are therefore left with showing that that the rectangle formed by the first three terms in the sequences commutes. Schematically, this is the equation:

$$\eta \circ (\eta\eta) \circ (\mathrm{id}s\mathrm{id}) = (ss) \circ \eta \circ (\eta\eta).$$

We would like to use the fact that η satisfies the condition from Definition 3.3, but for this we are using the symmetry on exactly the wrong factors. However, because η respects the associators, we can replace

$$\eta \circ (\eta\eta) = \eta \circ (\mathrm{id}\eta) \circ (\mathrm{id}\eta\mathrm{id}),$$

to see

$$\begin{aligned}\eta \circ (\eta\eta) \circ s &= \eta \circ (\mathrm{id}\eta) \circ (\mathrm{id}\eta\mathrm{id}) \circ (\mathrm{id}s\mathrm{id}) = \eta \circ (\mathrm{id}\eta) \circ (\mathrm{id}ss\mathrm{id}) \circ (\mathrm{id}\eta\mathrm{id}) \\ &= (ss) \circ \eta \circ (\eta\eta),\end{aligned}$$

where the second equality is the compatibility of η with the symmetry, and the final equality is the naturality of η. Similarly, ζ is compatible with the symmetry and the associators, and so Z is a functor. □

Because $\eta \circ \zeta = \mathrm{id}$, the functor Z is in fact a right sided inverse to H.

4.2.3 $\mathcal{Z}(\mathcal{A})$-crossed categories

We can now define the notions of $\mathcal{Z}(\mathcal{A})$-crossed tensor and braided categories.

$\mathcal{Z}(\mathcal{A})$-crossed monoidal categories

Definition 4.16. Let $\mathcal{K}$ be a $\mathcal{Z}(\mathcal{A})_s$-enriched and tensored category. A *$\mathcal{Z}(\mathcal{A})$-crossed tensor structure on $\mathcal{K}$* is a functor:

$$\otimes\colon \mathcal{K} \underset{c}{\boxtimes} \mathcal{K} \to \mathcal{K},$$

together with a functor

$$\mathbb{I}\colon \mathcal{A}_{\mathcal{Z}} \to \mathcal{K},$$

and associators and unitors that satisfy the usual coherence conditions. If $\mathcal{K}$ is rigid and $\mathbb{I}(\mathbb{I}_{\mathcal{A}})$ is a simple object of $\mathcal{K}$, we will call it *$\mathcal{Z}(\mathcal{A})$-crossed fusion*.

Proposition 4.17. *Let $\mathcal{K}$ and $\mathcal{L}$ be $\mathcal{Z}(\mathcal{A})$-crossed tensor categories. Then $\mathcal{K} \underset{s}{\boxtimes} \mathcal{L}$ is $\mathcal{Z}(\mathcal{A})$-crossed tensor, with monoidal structure given by the componentwise tensor product.*

Proof. The componentwise tensor product is given by the composite of Z from Proposition 4.15 with the image under the change of basis along $\otimes_s$ of the $\mathcal{Z}(\mathcal{A})_s \boxtimes \mathcal{Z}(\mathcal{A})_s$-enriched functor

$$\mathcal{K} \underset{c}{\boxtimes} \mathcal{K} \boxtimes \mathcal{L} \underset{c}{\boxtimes} \mathcal{L} \to \mathcal{K} \boxtimes \mathcal{L}.$$

As these are $\mathcal{Z}(\mathcal{A})$-enriched functors, so is the componentwise tensor product. To establish associativity and unitality, we observe that Z is compatible with the associators and unitors for $\mathcal{Z}(\mathcal{A})$ and hence will preserve the componentwise associators. $\square$

Lemma 4.18. *Let $\mathcal{K}$ and $\mathcal{L}$ be $\mathcal{Z}(\mathcal{A})$-crossed tensor categories. Then the switch map $S\colon \mathcal{K} \underset{s}{\boxtimes} \mathcal{L} \to \mathcal{L} \underset{s}{\boxtimes} \mathcal{K}$, that uses the symmetry in $\mathcal{Z}(\mathcal{A})_s$ on hom-objects is a monoidal functor, in the sense that*

$$\begin{tikzcd}
\mathcal{K} \underset{s}{\boxtimes} \mathcal{L} \underset{c}{\boxtimes} \mathcal{K} \underset{s}{\boxtimes} \mathcal{L} \arrow[r, "\otimes"] \arrow[d, "S \underset{c}{\boxtimes} S"] & \mathcal{K} \underset{s}{\boxtimes} \mathcal{L} \arrow[d, "S"] \\
(\mathcal{L} \underset{s}{\boxtimes} \mathcal{K}) \underset{c}{\boxtimes} (\mathcal{L} \underset{s}{\boxtimes} \mathcal{K}) \arrow[r, "\otimes"] & \mathcal{L} \underset{s}{\boxtimes} \mathcal{K}
\end{tikzcd}$$

commutes up to natural isomorphism.

Proof. We will show that in this case the diagram strictly commutes. Unpacking the definition of the $\mathcal{Z}(\mathcal{A})$-crossed monoidal structure on $\mathcal{K} \underset{s}{\boxtimes} \mathcal{L}$, we get

$$\begin{tikzcd}
\mathcal{K} \underset{s}{\boxtimes} \mathcal{L} \underset{c}{\boxtimes} \mathcal{K} \underset{s}{\boxtimes} \mathcal{L} \arrow[r, "Z"] \arrow[d, "S \underset{c}{\boxtimes} S"] & \mathcal{K} \underset{c}{\boxtimes} \mathcal{K} \underset{s}{\boxtimes} \mathcal{L} \underset{c}{\boxtimes} \mathcal{L} \arrow[r, "\otimes_{\mathcal{K}} \underset{s}{\boxtimes} \otimes_{\mathcal{L}}"] \arrow[d, "S"] & \mathcal{K} \underset{s}{\boxtimes} \mathcal{L} \arrow[d, "S"] \\
(\mathcal{L} \underset{s}{\boxtimes} \mathcal{K}) \underset{c}{\boxtimes} (\mathcal{L} \underset{s}{\boxtimes} \mathcal{K}) \arrow[r, "Z"] & \mathcal{L} \underset{c}{\boxtimes} \mathcal{L} \underset{s}{\boxtimes} \mathcal{K} \underset{c}{\boxtimes} \mathcal{K} \arrow[r, "\otimes_{\mathcal{K}} \underset{s}{\boxtimes} \otimes_{\mathcal{L}}"] & \mathcal{L} \underset{s}{\boxtimes} \mathcal{K}.
\end{tikzcd}$$

The leftmost square commutes as a direct consequence of ζ being compatible with the symmetry, as it satisfies Definition 3.3. The rightmost square commutes as the top route is change of basis along $\otimes_s$ for $\otimes_{\mathcal{K}} \boxtimes \otimes_{\mathcal{L}}$ composed with the switch functor, whereas the bottom is change of basis along $\otimes_s$ composed with the switch map on the same functor, and the symmetry is a natural isomorphism between these change of basis functors. ☐

Similarly to Proposition A.27, we have:

Lemma 4.19. *Let $\mathcal{K}$ be $\mathcal{Z}(\mathcal{A})$-crossed tensor, then $\overline{\mathcal{K}}$ is $\mathcal{A}$-tensor.*

Proof. We have already established that $\overline{\mathcal{K} \underset{c}{\boxtimes} \mathcal{K}} = \overline{\mathcal{K}} \underset{\mathcal{A}}{\boxtimes} \overline{\mathcal{K}}$ in Lemma 4.10, this means that the image under change of basis along the forgetful functor of the $\mathcal{Z}(\mathcal{A})$-crossed monoidal structure is a functor $\overline{\mathcal{K}} \underset{\mathcal{A}}{\boxtimes} \overline{\mathcal{K}} \to \overline{\mathcal{K}}$. Furthermore, $\overline{\mathcal{A}_{\mathcal{Z}}} = \underline{\mathcal{A}}$, the category $\mathcal{A}$ enriched over itself, so the image of the unit for $\mathcal{K}$ is a functor $\underline{\mathcal{A}} \to \overline{\mathcal{K}}$, as required. By the 2-functoriality of the change of basis along the forgetful functor, the images of the unitor and associator will act as unitors and associators for $\overline{\mathcal{K}}$. ☐

$\mathcal{Z}(\mathcal{A})$-crossed braided categories

Definition 4.20. Let $\mathcal{K}$ be $\mathcal{Z}(\mathcal{A})$-crossed tensor. Then a *crossed braiding* for $\mathcal{K}$ is a natural isomorphism between $\otimes \colon \mathcal{K} \underset{c}{\boxtimes} \mathcal{K} \to \mathcal{K}$ and

$$\otimes \colon \mathcal{K} \underset{c}{\boxtimes} \mathcal{K} \xrightarrow{B} \mathcal{K} \underset{c}{\boxtimes} \mathcal{K} \xrightarrow{\otimes} \mathcal{K},$$

that satisfies the hexagon equations.

Proposition 4.21. *Let $\mathcal{K}$ and $\mathcal{L}$ be $\mathcal{Z}(\mathcal{A})$-crossed braided categories. Then $\mathcal{K} \underset{s}{\boxtimes} \mathcal{L}$ is $\mathcal{Z}(\mathcal{A})$-crossed braided.*

Proof. We will show that the componentwise braiding is compatible with the componentwise $\mathcal{Z}(\mathcal{A})$-crossed monoidal structure. That is,

$$\begin{tikzcd}
\mathcal{K} \underset{s}{\boxtimes} \mathcal{L} \underset{c}{\boxtimes} \mathcal{K} \underset{s}{\boxtimes} \mathcal{L} \arrow[r, "Z"] \arrow[d, "B"] & \mathcal{K} \underset{c}{\boxtimes} \mathcal{K} \underset{s}{\boxtimes} \mathcal{L} \underset{c}{\boxtimes} \mathcal{L} \arrow[r, "\otimes_{\mathcal{K}} \underset{s}{\boxtimes} \otimes_{\mathcal{L}}"] \arrow[d, "B \underset{s}{\boxtimes} B"] & \mathcal{K} \underset{s}{\boxtimes} \mathcal{L} \arrow[d, "="] \\
\mathcal{K} \underset{s}{\boxtimes} \mathcal{L} \underset{c}{\boxtimes} \mathcal{K} \underset{s}{\boxtimes} \mathcal{L} \arrow[r, "Z"] & \mathcal{K} \underset{c}{\boxtimes} \mathcal{K} \underset{s}{\boxtimes} \mathcal{L} \underset{c}{\boxtimes} \mathcal{L} \arrow[r, "\otimes_{\mathcal{K}} \underset{s}{\boxtimes} \otimes_{\mathcal{L}}"] & \mathcal{K} \underset{s}{\boxtimes} \mathcal{L}
\end{tikzcd}$$

commutes up to componentwise braiding. The leftmost square commutes as a consequence of ζ satisfying 3.3, the rightmost square commutes up to the $\underset{s}{\boxtimes}$ product of the braidings for $\mathcal{K}$ and $\mathcal{L}$. ☐

Lemma 4.22. *The switch functor S from Lemma 4.18 is braided monoidal.*

Proof. We have to check that S takes the componentwise braiding to the componentwise braiding. This is immediate from the symmetry and the braiding commuting with each other in $\mathcal{Z}(\mathcal{A})$. ☐

The Neutral Subcategory

In a $\mathcal{Z}(\mathcal{A})$-crossed braided category $\mathcal{K}$, the full subcategory $\mathcal{K}_{\mathcal{A}}$ of objects for which the hom-objects are contained in the subcategory $\mathcal{A} \subset \mathcal{Z}(\mathcal{A})$ forms an braided $\mathcal{A}$-tensor category: the functor B restricts to the switch functor for the $\mathcal{A}$-product here, and by Proposition 2.26 the symmetric tensor product is the tensor product of $\mathcal{A}$ on this subcategory. We can define $\mathcal{K}_{\mathcal{A}}$ by:

Definition 4.23. Let $\mathcal{K}$ be a $\mathcal{Z}(\mathcal{A})$-crossed braided category. Then the full subcategory on objects $k \in \mathcal{K}$ for which the Yoneda embedding factors as:

$$\mathcal{K}(-,k) \colon \mathcal{K}^{\mathrm{op}} \to \mathcal{A} \hookrightarrow \mathcal{Z}(\mathcal{A}),$$

is called the *neutral subcategory* of $\mathcal{K}$ and will be denoted by $\mathcal{K}_{\mathcal{A}}$.

There is another characterisation of the neutral subcategory:

Lemma 4.24. *Let k be an object of a $\mathcal{Z}(\mathcal{A})_s$-enriched category $\mathcal{K}$. This object is in $\mathcal{K}_{\mathcal{A}}$ if and only if the endomorphism object $\mathcal{K}(k,k)$ of k is an object of $\mathcal{A} \subset \mathcal{Z}(\mathcal{A})$.*

Proof. The "only if" direction is obvious. For the other direction, observe that for any $k' \in \mathcal{K}$, we have an automorphism of $\mathcal{K}(k',k)$ given by the composite

$$\mathcal{K}(k',k) \cong \mathbb{I}_s \otimes_s \mathcal{K}(k',k) \xrightarrow{\mathrm{id}_k} \mathcal{K}(k,k) \otimes_s \mathcal{K}(k',k) \xrightarrow{\circ} \mathcal{K}(k',k).$$

Assuming that $\mathcal{K}(k,k) \in \mathcal{A} \subset \mathcal{Z}(\mathcal{A})$, we see, by Proposition 2.26, that this automorphism factors through an object of $\mathcal{A}$, and hence that $\mathcal{K}(k',k)$ is an object of $\mathcal{A}$. $\square$

By Proposition 2.26, the subcategory $\mathcal{A} \subset \mathcal{Z}(\mathcal{A})$ is "orthogonal" to its complement. This translates to the following for the product $\underset{s}{\boxtimes}$ of $\mathcal{Z}(\mathcal{A})_s$-enriched and tensored categories from Definition 4.7:

Proposition 4.25. *Let $\mathcal{K}$ and $\mathcal{L}$ be $\mathcal{Z}(\mathcal{A})_s$-enriched and tensored categories. Then:*

$$(\mathcal{K} \underset{s}{\boxtimes} \mathcal{L})_{\mathcal{A}} \cong \mathcal{K} \underset{s}{\boxtimes} \mathcal{L}_{\mathcal{A}} \cong \mathcal{K}_{\mathcal{A}} \underset{s}{\boxtimes} \mathcal{L} \cong \overline{\mathcal{K}_{\mathcal{A}}} \underset{\mathcal{A}}{\boxtimes} \overline{\mathcal{L}_{\mathcal{A}}},$$

where we view the $\mathcal{A}$-enriched and tensored category on the right as $\mathcal{Z}(\mathcal{A})_s$-enriched and tensored category by using the symmetric strictly monoidal inclusion functor $\mathcal{A} \hookrightarrow \mathcal{Z}(\mathcal{A})$.

Proof. We prove the equivalence between the left- and rightmost categories first. The category on the left hand side is a full subcategory of $\mathcal{K} \underset{s}{\boxtimes} \mathcal{L}$, we define a functor $\overline{\mathcal{K}_{\mathcal{A}}} \underset{\mathcal{A}}{\boxtimes} \overline{\mathcal{L}_{\mathcal{A}}} \to \mathcal{K} \underset{s}{\boxtimes} \mathcal{L}$ by using the inclusion $\mathcal{A} \hookrightarrow \mathcal{Z}(\mathcal{A})$ on hom-objects and claim its essential image is $(\mathcal{K} \underset{s}{\boxtimes} \mathcal{L})_{\mathcal{A}}$.

To show this functor is essentially surjective onto $(\mathcal{K} \underset{s}{\boxtimes} \mathcal{L})_{\mathcal{A}}$, let $k \boxtimes l$ be an object of $(\mathcal{K} \underset{s}{\boxtimes} \mathcal{L})_{\mathcal{A}}$. For this k and l, denote the summands contained in $\mathcal{K}_{\mathcal{A}}$ and $\mathcal{L}_{\mathcal{A}}$ by $k_{\mathcal{A}}$ and $l_{\mathcal{A}}$, respectively. We claim that:

$$k_{\mathcal{A}} \boxtimes l_{\mathcal{A}} \cong k \boxtimes l.$$

We will show this by examining their Yoneda embeddings. The object on the left hand side has Yoneda embedding $\mathcal{K}(-, k_{\mathcal{A}}) \otimes_s \mathcal{L}(-, l_{\mathcal{A}})$ whereas the right hand side has $\mathcal{K}(-, k) \otimes_s \mathcal{L}(-, l)$, and we claim that the image of the inclusions $i_k : k_{\mathcal{A}} \hookrightarrow k$ and $i_l : l_{\mathcal{A}} \hookrightarrow l$ is a natural isomorphism between these functors. Let $k' \boxtimes l'$ be an object of $\mathcal{K} \underset{s}{\boxtimes} \mathcal{L}$, then we want to show that

$$\mathcal{K}(k', k_{\mathcal{A}}) \otimes_s \mathcal{L}(l', l_{\mathcal{A}}) \xrightarrow{(i_k)_* \otimes_s (i_l)_*} \mathcal{K}(k', k) \otimes_s \mathcal{L}(l', l)$$

is an isomorphism. By Proposition 2.26, the symmetric tensor product of two objects in $\mathcal{Z}(\mathcal{A})$ is a non-zero object of $\mathcal{A}$ if and only if the $\mathcal{A}$-summands of these objects are non-zero, and the part that lies in $\mathcal{A}$ is the product of these summands. The objects $\mathcal{K}(k', k_{\mathcal{A}})$ and $\mathcal{L}(l', l_{\mathcal{A}})$ are the $\mathcal{A}$ summands of $\mathcal{K}(k', k_{\mathcal{A}})$ and $\mathcal{L}(l', l_{\mathcal{A}})$ respectively, so the claim follows. The same argument also establishes the functor is fully faithful.

To see the other equivalences, note that the argument above also works if we only take the neutral summand of k or l. □

A 2-category of $\mathcal{Z}(\mathcal{A})$-crossed braided categories

For future reference, it will be useful to define the following:

Definition 4.26. The *symmetric monoidal 2-category* $\mathcal{Z}(\mathcal{A})$-**XBF** *of* $\mathcal{Z}(\mathcal{A})$-*crossed braided fusion categories* is the 2-category with

- objects $\mathcal{Z}(\mathcal{A})$-crossed braided fusion categories,
- morphisms braided monoidal $\mathcal{Z}(\mathcal{A})_s$-enriched functors $F : \mathcal{C} \to \mathcal{C}'$, together with a natural transformation μ with components for $a \in \mathcal{Z}(\mathcal{A})$ and $c \in \mathcal{C}$
$$F(ac) \xrightarrow[\cong]{\mu_{a,c}} aF(c),$$
such that for all $a \in \mathcal{Z}(\mathcal{A})$ and $c, c' \in \mathcal{C}$, the following diagram commutes:
$$\begin{tikzcd} \mathcal{C}(c, ac') \arrow[rr, "\cong"] \arrow[d, "F"] & & a\mathcal{C}(c, c') \arrow[d, "\mathrm{id}_a \otimes F"] \\ \mathcal{C}'(F(c), F(ac')) \arrow[r, "\mu"] & \mathcal{C}'(F(c), aF(c')) & a\mathcal{C}(F(c), F(c')), \arrow[l, "\cong"'] \end{tikzcd}$$
- and 2-morphisms monoidal natural transformations, that make the diagrams

$$\begin{tikzcd}
F(ac) \arrow[r, "\eta_{ac}"] \arrow[d, "\cong"] & G(ac) \arrow[d, "\cong"] \\
a \cdot F(c) \arrow[r, "\mathrm{id}_a \cdot \eta_c"] & a \cdot G(c),
\end{tikzcd}$$

commute.

4.3 (Super)-Tannakian case

This part of this Chapter is devoted to proving:

Theorem 4.27. *Let G (or (G, ω)) be a finite (super) group. Then the functor $\overline{\overline{(-)}}$ (see Definition 4.38 and Section 4.3.2) from $\mathcal{Z}(\mathcal{A})$-**XBF** (Definition 4.26) to G-**XBF** (or (G, ω)-**XBF**) (Definition 4.34) is a symmetric monoidal equivalence, with inverse given by **Fix** (see Definitions 4.52, 4.56 and 4.57).*

We will start by introducing the relevant notions in Section 4.3.1, in Section 4.3.2 we show how define the functor $\overline{\overline{(-)}}$ and give an item by item proof that it lands in G-crossed braided categories. This the content of Theorem 4.36. In Section 4.3.3 we show to define **Fix** to produce from a G-crossed braided category a $\mathcal{Z}(\mathcal{A})$-crossed tensor category.

4.3.1 Preliminaries

Notation 4.28. Throughout the remainder of this chapter, we will fix a choice Φ of fibre functor on our symmetric fusion category $\mathcal{A}$.

The Drinfeld Centre as G-equivariant vector bundles

With Theorem A.35 in hand, we can view $\mathcal{A}$ as representations of some finite (super)-group. The Drinfeld centre of the category of representations of a finite group G is well-known [BK01, Chapter 3.2] to be (braided monoidal) equivalent to the category of vector bundles over G, equivariant for the conjugation action of G on itself. This result extends to the super-group case, see Section A.2.3. Additionally, it was shown in Chapter 2 that the symmetric tensor product on the Drinfeld centre agrees with the (graded) fibrewise tensor product. We will make use of the following facts. In this model for the Drinfeld centre, the forgetful functor $\mathcal{Z}(\mathcal{A}) \to \mathcal{A}$ is given by summing over the fibres. Applying the fibre functor Φ to this produces a vector space, which carries a G-grading by remembering over which elements the fibres sat. Additionally, this vector space carries a G-action which conjugates the grading.

G-crossed braided categories

Definition 4.29 ([Tur10])**.** A *G-crossed braided fusion category* is a **Vect**-enriched and tensored category $\mathcal{C}$, together with:

(i) for each $g \in G$ a **Vect**-enriched and tensored semi-simple category $\mathcal{C}_g$ with finitely many simples, decomposing $\mathcal{C}$ as $\mathcal{C} = \oplus_{g\in G}\mathcal{C}_g$ (a *G-graded linear category*);

(ii) a *G-graded fusion structure*: a tensor structure $\otimes\colon \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$, such that $\otimes\colon \mathcal{C}_g \boxtimes \mathcal{C}_h \to \mathcal{C}_{gh}$, that is rigid with a simple unit;

(iii) a homomorphism $G \to (\mathrm{Aut}(\mathcal{C}))$. The image of $g \in G$ under this homomorphism will be denoted $(-)^g$. We require $(-)^g\colon \mathcal{C}_h \to \mathcal{C}_{ghg^{-1}}$. (This is called a *G-crossing.*)

(iv) a *crossed braiding*: for each $g \in G$ a natural isomorphism between $\otimes\colon \mathcal{C}_g \boxtimes \mathcal{C} \to \mathcal{C}$ and
$$\mathcal{C}_g \boxtimes \mathcal{C} \xrightarrow{S} \mathcal{C} \boxtimes \mathcal{C}_g \xrightarrow{(-)^g\boxtimes\mathrm{Id}} \mathcal{C} \boxtimes \mathcal{C}_g \xrightarrow{\otimes} \mathcal{C},$$
satisfying the hexagon equations.

Super G-crossed braided categories

We will now introduce the notion of G-crossed braided category, which the author believes to be new. Before we give the definition, we need to following.

Definition 4.30. Let $\mathcal{C}$ be a **sVect**-enriched category. Then the *grading involution functor* Π on $\mathcal{C}$ is the autofunctor on $\mathcal{C}$ that acts as the identity on objects and even morphisms, and as $-\mathrm{id}$ on odd morphisms.

Definition 4.31. A *super G-crossed braided category* is an **sVect**-enriched and tensored category $\mathcal{C}$, together with:

(i) for each $g \in G$ an **sVect**-enriched and tensored category $\mathcal{C}_g$ that is semi-simple with finitely many simples, decomposing $\mathcal{C}$ as $\mathcal{C} = \oplus_{g\in G}\mathcal{C}_g$ (a *G-graded super linear category*);

(ii) a G-graded super fusion stucture: a super tensor structure $\otimes\colon \mathcal{C} \underset{\mathbf{sVect}}{\boxtimes} \mathcal{C} \to \mathcal{C}$, such that $\otimes\colon \mathcal{C}_g \underset{\mathbf{sVect}}{\boxtimes} \mathcal{C}_h \to \mathcal{C}_{gh}$ that is rigid and has a simple unit object;

(iii) a homomorphism $(G, \omega) \to (\mathrm{Aut}(\mathcal{C}), \Pi)$ of pointed groups, where Π denotes the grading involution functor. The image of $g \in G$ under this homomorphism will be denoted $(-)^g$. We require $(-)^g\colon \mathcal{C}_h \to \mathcal{C}_{ghg^{-1}}$. (This is called a *super G-crossing.*)

(iv) a *crossed braiding*, for each $g \in G$ a natural isomorphism between $\otimes\colon \mathcal{C}_g \underset{\mathbf{sVect}}{\boxtimes} \mathcal{C} \to \mathcal{C}$ and
$$\mathcal{C}_g \underset{\mathbf{sVect}}{\boxtimes} \mathcal{C} \xrightarrow{S} \mathcal{C} \underset{\mathbf{sVect}}{\boxtimes} \mathcal{C}_g \xrightarrow{(-)^g\boxtimes\mathrm{Id}} \mathcal{C} \underset{\mathbf{sVect}}{\boxtimes} \mathcal{C}_g \xrightarrow{\otimes} \mathcal{C},$$
satisfying the hexagon equations.

The degreewise product of (super) G-crossed braided categories

Definition 4.32. Let $\mathcal{C}$ and $\mathcal{D}$ be G-graded (super) linear categories, then the *degreewise product of* $\mathcal{C}$ *and* $\mathcal{D}$ is defined by:

$$\mathcal{C} \underset{G}{\boxtimes} \mathcal{D} = \bigoplus_{g \in G} \mathcal{C}_g \boxtimes \mathcal{D}_g,$$

where in the super case we use $\underset{\mathbf{sVect}}{\boxtimes}$ instead of $\boxtimes$. Both these operations are a special case of Definition A.17.

Proposition 4.33. *The degreewise product* $\mathcal{C} \underset{G}{\boxtimes} \mathcal{D}$*, of* G*-crossed braided categories* $\mathcal{C}$ *and* $\mathcal{D}$*, is* G*-crossed braided, for the componentwise tensor product,* G*-crossing and crossed braiding.*

Proof. The category $\mathcal{C} \underset{G}{\boxtimes} \mathcal{D}$ is G-graded by construction. The componentwise tensor product is given by:

$$\mathcal{C}_g \boxtimes \mathcal{D}_g \boxtimes \mathcal{C}_h \boxtimes \mathcal{D}_h \xrightarrow{\mathrm{Id} \boxtimes S \boxtimes \mathrm{Id}} \mathcal{C}_g \boxtimes \mathcal{D}_g \boxtimes \mathcal{C}_h \boxtimes \mathcal{D}_h \xrightarrow{\otimes \boxtimes \otimes} \mathcal{C}_{gh} \boxtimes \mathcal{D}_{gh},$$

where we use $\underset{\mathbf{sVect}}{\boxtimes}$ and its switch map instead of $\boxtimes$ in the super case. This clearly respects the G-grading.

Any pair of automorphisms of $\mathcal{C}$ and $\mathcal{D}$ that conjugate the grading will induce an automorphism of $\mathcal{C} \underset{G}{\boxtimes} \mathcal{D}$ that conjugates the grading. In the super case, we observe that the **sVect**-product of the $\mathbb{Z}_2$-grading involution on **sVect**-enriched categories is the grading involution on the product, so the G-crossings of $\mathcal{C}$ and $\mathcal{D}$ will give an homomorphism of pointed groups as desired by (iii) in Definition 4.31.

To see that the componentwise braiding gives a crossed braiding, observe that we need for $g \in G$ a natural isomorphism between the componentwise tensor product and

$$(\mathcal{C} \underset{G}{\boxtimes} \mathcal{D})_g \boxtimes (\mathcal{C} \underset{G}{\boxtimes} \mathcal{D}) \xrightarrow{S} (\mathcal{C} \underset{G}{\boxtimes} \mathcal{D}) \boxtimes (\mathcal{C} \underset{G}{\boxtimes} \mathcal{D})_g \xrightarrow{(-)_g \otimes \mathrm{Id}} (\mathcal{C} \underset{G}{\boxtimes} \mathcal{D}) \boxtimes (\mathcal{C} \underset{G}{\boxtimes} \mathcal{D})_g \xrightarrow{\otimes} (\mathcal{C} \underset{G}{\boxtimes} \mathcal{D}),$$

where in the super case we replace $\boxtimes$ with $\underset{\mathbf{sVect}}{\boxtimes}$. That is, for each $h \in G$, the following diagram should commute up to the braiding:

$$\begin{tikzcd}
\mathcal{C}_g \boxtimes \mathcal{D}_g \boxtimes \mathcal{C}_h \boxtimes \mathcal{D}_h \arrow[r, "S"] \arrow[d, "\mathrm{Id} \boxtimes S \boxtimes \mathrm{Id}"] & \mathcal{C}_h \boxtimes \mathcal{D}_h \boxtimes \mathcal{C}_g \boxtimes \mathcal{D}_g \arrow[d, "(-)_g \boxtimes (-)_g \boxtimes \mathrm{Id} \boxtimes \mathrm{Id}"] \\
\mathcal{C}_g \boxtimes \mathcal{C}_h \boxtimes \mathcal{D}_g \boxtimes \mathcal{D}_h \arrow[d, "\otimes \boxtimes \otimes"] & \mathcal{C}_{ghg^{-1}} \boxtimes \mathcal{D}_{ghg^{-1}} \boxtimes \mathcal{C}_g \boxtimes \mathcal{D}_g \arrow[d, "\mathrm{Id} \boxtimes S \boxtimes \mathrm{Id}"] \\
\mathcal{C}_{gh} \boxtimes \mathcal{D}_{gh} & \mathcal{C}_{ghg^{-1}} \boxtimes \mathcal{C}_g \boxtimes \mathcal{D}_{ghg^{-1}} \boxtimes \mathcal{D}_g. \arrow[l, "\otimes \boxtimes \otimes"']
\end{tikzcd}$$

Using that the switch map is natural, we can exchange the maps along the right hand side to get:

$$\begin{tikzcd}
\mathcal{C}_g \boxtimes \mathcal{D}_g \boxtimes \mathcal{C}_h \boxtimes \mathcal{D}_h \arrow[r, "S"] \arrow[d, "\mathrm{Id}\boxtimes S\boxtimes\mathrm{Id}"] & \mathcal{C}_h \boxtimes \mathcal{D}_h \boxtimes \mathcal{C}_g \boxtimes \mathcal{D}_g \arrow[d, "\mathrm{Id}\boxtimes S\boxtimes\mathrm{Id}"] \\
\mathcal{C}_g \boxtimes \mathcal{C}_h \boxtimes \mathcal{D}_g \boxtimes \mathcal{D}_h \arrow[r, "S\boxtimes S"] \arrow[d, "\otimes\boxtimes\otimes"] & \mathcal{C}_h \boxtimes \mathcal{C}_g \boxtimes \mathcal{D}_h \boxtimes \mathcal{D}_g \arrow[d, "(-)_g\boxtimes(-)_g\boxtimes\mathrm{Id}\boxtimes\mathrm{Id}"] \\
\mathcal{C}_{gh} \boxtimes \mathcal{D}_{gh} & \mathcal{C}_{ghg^{-1}} \boxtimes \mathcal{D}_{ghg^{-1}} \boxtimes \mathcal{C}_g \boxtimes \mathcal{D}_g . \arrow[l, "\otimes\boxtimes\otimes"']
\end{tikzcd}$$

The top square commutes strictly, and the bottom square indeed commutes up to the product of the braidings. ☐

A 2-category of (super) G-crossed braided fusion categories

The (super) G-crossed categories fit into a symmetric monoidal 2-category:

Definition 4.34. Let G (or (G,ω)) be a (super) group. Then the symmetric monoidal 2-category G-**XBF** (or (G,ω)-**XBF**) has objects G- (or (G,ω)-)crossed braided fusion categories. The 1-morphisms are (super) linear braided monoidal functors $F\colon \mathcal{C} \to \mathcal{C}'$, satisfying $F(\mathcal{C}_g) \subset \mathcal{C}_g$ and $F \circ (-)^g = (F(-))^g$ for all $g \in G$. The 2-morphisms are monoidal natural transformations κ satisfying $(\kappa_c)^g = \kappa_{c^g}$. The symmetric monoidal structure is given by the degreewise tensor product, with switch map given by the degreewise switch map of (super)-linear categories.

Remark 4.35. The definitions of (super) G-crossed braided category and functors between them used here are strict. One can also consider G-actions that are 2-functors from the 2-category with one object and no non-trivial 2-morphisms G to the 2-category with one object $\mathrm{Aut}(\mathcal{C})$, and allow functors to preserve the G-action up to natural isomorphism.

4.3.2 From $\mathcal{Z}(\mathcal{A})$-crossed to (super) G-crossed

In this section we will explain how to produce from a $\mathcal{Z}(\mathcal{A})$-crossed braided fusion category a (super) G-crossed braided fusion category:

Proposition 4.36. *Let $\mathcal{A} = \mathrm{Rep}(G)$ (or $\mathrm{Rep}(G,\omega)$). For any $\mathcal{K}$ be a $\mathcal{Z}(\mathcal{A})$-crossed braided fusion category, the (super) linear category $\overline{\overline{\mathcal{K}}}$ obtained from $\mathcal{K}$ (see Definition 4.38) is (super) G-crossed braided fusion (see Definitions 4.29 and 4.31).*

After this, in Section 4.3.2, we will show how to extend this to a 2-functor from $\mathcal{Z}(\mathcal{A})$-**XBF** to G-**XBF** (or (G,ω)-**XBF**).

The induced map for the fibre functor

Given a $\mathcal{Z}(\mathcal{A})_s$-enriched category $\mathcal{K}$, we can produce a (super) linear category out of $\overline{\mathcal{K}}$ (Definition 4.2) by changing basis along the fibre functor Φ for $\mathcal{A}$.

Lemma 4.37. *Let $\mathcal{K}$ be a $\mathcal{Z}(\mathcal{A})_s$-enriched and tensored category. Then $\Phi\overline{\mathcal{K}}$ is a (super)-linear category.*

Proof. As the fibre functor is monoidal, this is a direct consequence of Proposition A.31. □

The resulting category will usually not be idempotent complete, even when the original category was. We set:

Definition 4.38. Let $\mathcal{K}$ be a $\mathcal{Z}(\mathcal{A})_s$-enriched and tensored category. Then *the (super) linearisation $\overline{\overline{\mathcal{K}}}$ of $\mathcal{K}$* is the idempotent completion of $\Phi\overline{\mathcal{K}}$.

We observe that, as the fibre functor is unique up to monoidal natural isomorphism [Del90, Del02], the category $\overline{\overline{\mathcal{K}}}$ is unique up to equivalence.

G-grading

Since, on $\mathbf{Vect}_G[G]$, the forgetful functor followed by the fibre functor takes objects to G-graded (super) vector spaces, the morphisms of $\Phi\overline{\mathcal{K}}$ are G-graded (super) vector spaces. This will induce a grading on the idempotents:

Lemma 4.39. *For every $\mathcal{Z}(\mathcal{A})_s$-enriched and tensored category $\mathcal{K}$, every minimal idempotent of $\Phi\overline{\mathcal{K}}$ is homogeneous for the G-grading on the hom-objects.*

Proof. Let k be an object of $\Phi\overline{\mathcal{K}}$. Composition of endomorphisms of k factors through the image of the symmetric tensor product of $\mathcal{K}(k,k)$ with itself. This image is the fibrewise (super) tensor product (Definitions 2.29 and 2.33, Theorem 2.35). Observe that an idempotent is necessarily even. In the super-case, the fibrewise super tensor product reduces to the fibrewise tensor product for even objects. Decomposing an even endomorphism ψ into homogeneous components ψ_g, the condition for ψ to be an idempotent becomes:

$$\psi \circ \psi = \sum_{g\in G} \psi_g \circ \psi_g = \sum_{g\in G} \psi_g = \psi,$$

which is a condition for each ψ_g separately. So ψ is idempotent if and only if all its homogeneous components are. In particular, any minimal idempotent is homogeneous. □

This means that there is a function from the simple objects of the category $\overline{\overline{\mathcal{K}}}$ to G. We would like to extend this to a direct sum decomposition of our category, so we need to establish:

Lemma 4.40. *Let k and k' be simple objects of $\overline{\overline{\mathcal{K}}}$ of degrees g and g', respectively. Then $\overline{\overline{\mathcal{K}}}(k,k')$ is zero unless $g = g'$.*

Furthermore, assume that $k = f_k \in \Phi\overline{\mathcal{K}}(\bar{k},\bar{k})$ and $k' = f'_k \in \Phi\overline{\mathcal{K}}(\bar{k}',\bar{k}')$ for objects $\bar{k},\bar{k}' \in \Phi\overline{\mathcal{K}}$ and idempotents f_k, f'_k. Then, denoting by $\Phi\overline{\mathcal{K}}(\bar{k},\bar{k}')_{g,p}$ taking the even ($p=0$) or odd ($p=1$) part of the summand $\Phi\overline{\mathcal{K}}(\bar{k},\bar{k}')_g$, any morphism of parity p between k and k' arises from composing a morphism in $\Phi\overline{\mathcal{K}}(\bar{k},\bar{k}')_{\omega^p g,p}$ with the idempotents.

Proof. In an idempotent completion, the hom-object between two objects is computed by pre- and post-composing with the idempotents. Composition factors over the fibrewise (super) tensor product, so morphisms between k and k' are in the image of the composition map out of:

$$\Phi\overline{\mathcal{K}}(\bar{k}', \bar{k}')_0^{g'} \otimes_f^{\omega} \Phi\overline{\mathcal{K}}(\bar{k}, \bar{k}') \otimes_f^{\omega} \Phi\overline{\mathcal{K}}(\bar{k}, \bar{k}')_0^{g},$$

where $\otimes_f^{\omega} = \otimes_f$ in the non-super case, and $\Phi\overline{\mathcal{K}}(\bar{k}', \bar{k}')_0^{g'}$ denotes the bundle supported by $\{g'\}$ with fibre $\Phi\overline{\mathcal{K}}(\bar{k}', \bar{k}')_{g',0}$. We have used similar notation for the rightmost factor. Computing the rightmost product, we see this is the bundle with fibres:

$$(\Phi\overline{\mathcal{K}}(\bar{k}, \bar{k}') \otimes_f^{\omega} \Phi\overline{\mathcal{K}}(\bar{k}, \bar{k}')_0^{g})_{h,p} = \Phi\overline{\mathcal{K}}(\bar{k}, \bar{k}')_{h,p} \otimes (\Phi\overline{\mathcal{K}}(\bar{k}, \bar{k}')_0^{g})_{\omega^p h}.$$

For this to be non-zero, we need $h = \omega^p g$, proving the "furthermore" part of the Lemma. Taking the fibrewise (super) tensor product of this with $\Phi\overline{\mathcal{K}}(\bar{k}', \bar{k}')^{g'}$, we get:

$$\begin{aligned}(\Phi\overline{\mathcal{K}}(\bar{k}', \bar{k}')_0^{g'} \otimes_f^{\omega} \Phi\overline{\mathcal{K}}(\bar{k}, \bar{k}') \otimes_f^{\omega} \Phi\overline{\mathcal{K}}(\bar{k}, \bar{k}')_0^{g})_{h',p'} &= \\ &(\Phi\overline{\mathcal{K}}(\bar{k}', \bar{k}')_0^{g'})_{\omega^{p'} h'} \otimes (\Phi\overline{\mathcal{K}}(\bar{k}, \bar{k}') \otimes_f^{\omega} \Phi\overline{\mathcal{K}}(\bar{k}, \bar{k}')_0^{g})_{h',p'}.\end{aligned}$$

We immediately see that for this to be non-zero requires $\omega^{p'} h' = g'$, and from the above computation $\omega^{p'} h' = g$, so we need $g = g'$ for this to be non-zero. □

Combining this with Lemma 4.39, we get:

Corollary 4.41. *The (super)-linear category $\overline{\overline{\mathcal{K}}}$ decomposes a direct sum*

$$\overline{\overline{\mathcal{K}}} = \bigoplus_{g \in G} \overline{\overline{\mathcal{K}}}_g$$

of (super)-linear categories.

We remind the reader that this corresponds to items (i) from Definition 4.29 and (i) from Definition 4.31.

Graded monoidal structure

Lemma 4.42. *$\overline{\overline{\mathcal{K}}}$ is (super)-fusion.*

Proof. This is a direct consequence of Proposition A.27 and Lemma 4.19. □

This (super-)tensor structure is graded in the sense that it satisfies item (ii) from Definition 4.29 (or item (ii) from Definition 4.31):

Lemma 4.43. *The (super)-tensor structure from Lemma 4.42 maps*

$$\overline{\overline{\mathcal{K}}}_g \boxtimes \overline{\overline{\mathcal{K}}}_h \to \overline{\overline{\mathcal{K}}}_{gh},$$

with respect to the decomposition from Corollary 4.41, and where we replace $\boxtimes$ with $\underset{\mathbf{sVect}}{\boxtimes}$ in the super case.

Proof. The grading of the (super-)tensor product of two homogeneous objects $k \in \overline{\overline{\mathcal{K}}}_g$ and $k' \in \overline{\overline{\mathcal{K}}}_h$ obtained by taking the tensor product of the idempotents in $\Phi\overline{K}$. This tensor product, in turn, factors over the convolution tensor product of G-graded (super) vector spaces, and this sends the g-graded and the h-graded part to the gh-graded part, so the (super-)tensor product of the idempotents will be homogeneous of degree gh. □

G-action

The G-graded vector spaces obtained by applying the forgetful and fibre functors to the objects of $\mathcal{Z}(\mathcal{A})$ carry an action of the (super)-group, that we will denote by $g\cdot$. This G-action conjugates the grading. This action of the (super-)group translates to an action on the idempotent completion:

Lemma 4.44. *Let $g \in G$, then the assignment*

$$\begin{aligned} (-)^g \colon \overline{\overline{\mathcal{K}}} &\to \overline{\overline{\mathcal{K}}} \\ k &\mapsto g \cdot k \\ \overline{\overline{\mathcal{K}}}(k, k') &\to g \cdot \overline{\overline{\mathcal{K}}}(g \cdot k, g \cdot k') \end{aligned}$$

defines an autofunctor of $\overline{\overline{\mathcal{K}}}$.

Proof. This assignment is clearly strictly invertible, with inverse given by $(-)^{g^{-1}}$, so we have to prove that it defines a functor. It is enough to show that the assignment

$$\begin{aligned} (-)^g \colon \Phi\overline{\mathcal{K}} &\to \Phi\overline{\mathcal{K}} \\ k &\mapsto k \\ \Phi\overline{\mathcal{K}}(k, k') &\to g \cdot \Phi\overline{\mathcal{K}}(k, k') \end{aligned}$$

is a functor, this will descend to the idempotent completion $\overline{\overline{\mathcal{K}}}$ as prescribed. As the identity morphisms are given by equivariant maps from $\mathbb{I}_s = \mathbb{C} \times G$ to the hom-objects, $(-)^g$ preserves identities. Recall that composition maps out of the fibrewise tensor product, and is a morphism in $\mathcal{Z}(\mathcal{A})$. Any morphism in $\mathcal{Z}(\mathcal{A})$ is a morphisms intertwining the G-action on the vector bundles over G, so we are trying to show that the fibrewise tensor product has the property that

$$g \cdot (V \otimes_f W) = (g \cdot V) \otimes_f (g \cdot W),$$

but this true by definition, see Definition 2.29. □

Lemma 4.45. *The assignment $g \mapsto (-)^g$ defines a homomorphism $G \to \mathrm{Aut}(\overline{\overline{\mathcal{K}}})$, or $(G, \omega) \to (\mathrm{Aut}(\overline{\overline{\mathcal{K}}}), \Pi)$ in the super case.*

Proof. In the non-super case, there is nothing to prove. In the super-case we observe that the $\mathbb{Z}_2$-grading on the G-equivariant vector bundles over G is exactly determined by whether ω acts by 1 or -1. So, $(-)^\omega$ will act by -1 exactly on the odd morphisms, i.e. as Π (Definition 4.30). □

We observe that this action of $g \in G$ on $\overline{\overline{\mathcal{K}}}$ takes $\overline{\overline{\mathcal{K}}}_h$ to $\overline{\overline{\mathcal{K}}}_{ghg^{-1}}$. This means that this action satisfies item (iii) from Definition 4.29 (or from Definition 4.31 in the super case).

G-crossed braiding

To prove Theorem 4.36, we still need to show that $\overline{\overline{\mathcal{K}}}$ satisfies (iv) from Definition 4.29 (or (iv) from Definition 4.31). The first step for this is:

Lemma 4.46. *The image of the functor B (see Definition 4.13) under the change of basis along the forgetful functor followed by Φ and then idempotent completion is given by:*

$$\overline{\overline{\mathcal{K}}}_g \boxtimes \overline{\overline{\mathcal{K}}} \xrightarrow{\text{Switch}} \overline{\overline{\mathcal{K}}} \boxtimes \overline{\overline{\mathcal{K}}}_g \xrightarrow{(-)^g \boxtimes \mathrm{Id}} \overline{\overline{\mathcal{K}}} \boxtimes \overline{\overline{\mathcal{K}}}_g, \tag{4.3}$$

where in the super case, we use $\underset{\mathbf{sVect}}{\boxtimes}$ instead of $\boxtimes$, and the switch map uses the symmetry in super vector spaces.

Proof. The image of the functor B (see Definition 4.13) under the change of basis along the forgetful functor followed by Φ is given by

$$\begin{aligned} \Phi\overline{B} \colon \Phi\overline{K} \boxtimes \Phi\overline{K} &\to \Phi\overline{K} \boxtimes \Phi\overline{K} \\ k \boxtimes k' &\mapsto k' \boxtimes k, \end{aligned}$$

and the image of the braiding on hom-objects. In the model of $\mathcal{Z}(\mathcal{A})$ as $\mathbf{Vect}_G[G]$, this braiding is given fibrewise by:

$$V_g \otimes W_h \to (g \cdot W_h) \otimes V_g.$$

Without loss of generality, let k_1, k_1' and k_2, k_2' be simple objects of $\overline{\overline{\mathcal{K}}}$ of degrees g and g', respectively. Then, by Lemma 4.40, morphisms $f \colon k_1 \to k_1'$ of parity p and $f' \colon k_2 \to k_2'$ of parity p' come from fibres over $\omega^p g$ or $\omega^{p'} g'$, respectively. This means that the image of B will take $f \otimes f'$ to $(\omega^p g \cdot f') \otimes f$, which, remembering that ω acts non-trivially only if $p' = 1$, we can rewrite as $(-1)^{pp'} g \cdot f' \otimes f$. But this is exactly what the composite from Equation (4.3) does. $\square$

Recall that the braiding for a $\mathcal{Z}(\mathcal{A})$-crossed braided category $\mathcal{K}$ is a natural isomorphism between $\otimes_{\mathcal{K}} \colon \mathcal{K} \underset{c}{\boxtimes} \mathcal{K} \to \mathcal{K}$ and the composite $\otimes_{\mathcal{K}} \circ B$. By Proposition A.24, this descends to a natural isomorphism between the images of these functors, so Lemma 4.46 has the following consequence:

Corollary 4.47. *The braiding for a $\mathcal{Z}(\mathcal{A})$-crossed braided fusion category $\mathcal{K}$ descends to a natural isomorphism between $\overline{\overline{\otimes}} \colon \overline{\overline{\mathcal{K}}}_g \boxtimes \overline{\overline{\mathcal{K}}} \to \overline{\overline{\mathcal{K}}}$ and*

$$\overline{\overline{\mathcal{K}}}_g \boxtimes \overline{\overline{\mathcal{K}}} \xrightarrow{\text{Switch}} \overline{\overline{\mathcal{K}}} \boxtimes \overline{\overline{\mathcal{K}}}_g \xrightarrow{(-)^g \boxtimes \mathrm{Id}} \overline{\overline{\mathcal{K}}} \boxtimes \overline{\overline{\mathcal{K}}}_g \xrightarrow{\overline{\overline{\otimes}}} \overline{\overline{\mathcal{K}}}.$$

Because this braiding satisfies coherence, so will its image. This shows that $\overline{\overline{\mathcal{K}}}$ satisfies item (iv) from Definition 4.29 (or item (iv) from Definition 4.31).

This completes the proof of Proposition 4.36.

The assignment $\overline{\overline{(-)}}$ is a 2-functor to G-XBF (or (G,ω)-XBF)

The aim of this subsection is to show that $\overline{\overline{(-)}}$ is a 2-functor from $\mathcal{Z}(\mathcal{A})$-**XBF** to G-**XBF** (or (G,ω)-**XBF** in the super group case). We first show that it takes functors of $\mathcal{Z}(\mathcal{A})$-crossed braided categories to functors on G-crossed braided fusion categories.

Proposition 4.48. *Let $F\colon \mathcal{K} \to \mathcal{K}'$ be a 1-morphism in $\mathcal{Z}(\mathcal{A})$-**XBF** (Definition 4.26). Then $\overline{\overline{F}}$ is a 1-morphism in G-**XBF** (or (G,ω)-**XBF** in the super group case).*

Proof. The fact that $\overline{\overline{F}}$ is a (super) linear braided monoidal functor is immediate from Proposition A.28 and the fact that F is braided monoidal. We still have to show that $\overline{\overline{F}}$ respects the direct sum decomposition of $\overline{\overline{\mathcal{K}}}$ and the G-action. For the former, observe that F acts by morphisms of $\mathcal{Z}(\mathcal{A})$ on the hom-objects. Viewing $\mathcal{Z}(\mathcal{A})$ as $\mathbf{Vect}_G[G]$, these morphisms are maps of vector bundles over G, so will descend to G-grading preserving morphisms, and will in particular send idempotents of degree g to idempotents of the same degree. Similarly, on hom-objects F will act by G-equivariant maps, this implies that $\overline{\overline{F}}$ will be G-equivariant. □

We also need that 2-morphisms in $\mathcal{Z}(\mathcal{A})$-**XBF** are sent to 2-morphisms in G-**XBF** (or (G,ω)-**XBF**).

Proposition 4.49. *Let κ be a 2-morphism in $\mathcal{Z}(\mathcal{A})$-**XBF** between $F, G\colon \mathcal{K} \to \mathcal{K}'$. Then $\overline{\overline{\kappa}}$ is a 2-morphism in G-**XBF** (or (G,ω)-**XBF** in the super group case.)*

Proof. It is clear that $\overline{\overline{\kappa}}$ will be monoidal. To see that it satisfies $(\overline{\overline{\kappa}}_k)^g = \overline{\overline{\kappa}}_{k^g}$ for each $k \in \overline{\overline{\mathcal{K}}}$ and $g \in G$, recall that a component κ_c of the enriched natural transformation is a morphism

$$\kappa_c\colon \mathbb{I}_s \to \mathcal{K}'(F(c), G(c)).$$

In $\mathbf{Vect}_G[G]$, we have $\mathbb{I}_s = \mathbb{C} \times G$, so κ_c is constant on each conjugacy class of G. Now, for $k \in \overline{\overline{\mathcal{K}}}$ homogeneous of degree h, the object k^g is homogeneous of degree $ghg{-}1$, that is, it comes from an idempotent of conjugate degree on the same object. But as $\overline{\overline{\kappa}}_k$ is defined by precomposing the image of κ with these idempotents under forget and fibre, this means that $\overline{\overline{\kappa}}$ satisfies the condition $(\overline{\overline{\kappa}}_k)^g = \overline{\overline{\kappa}}_{k^g}$. □

Degreewise tensor product

We now show that the assignment $\mathcal{K} \mapsto \overline{\overline{\mathcal{K}}}$ takes the product $\underset{s}{\boxtimes}$ (Definition 4.7) to the degreewise tensor product $\underset{G}{\boxtimes}$.

Proposition 4.50. *The 2-functor $\overline{\overline{(-)}}$ takes the enriched Cartesian product of $\mathcal{Z}(\mathcal{A})$-crossed braided fusion categories to the degreewise product of G-crossed braided fusion categories.*

This will be a consequence of:

Lemma 4.51. *Let $\mathcal{K}$ and $\mathcal{L}$ be $\mathcal{Z}(\mathcal{A})$-crossed fusion categories. Then*

$$\overline{\overline{\mathcal{K} \underset{s}{\boxtimes} \mathcal{L}}} = \overline{\overline{\mathcal{K}}} \underset{G}{\boxtimes} \overline{\overline{\mathcal{L}}}.$$

Proof. From Lemma 4.4, we know how to compare the forgetful image of the $\mathcal{Z}(\mathcal{A})_s$-enriched Cartesian product with the $\mathcal{A}$-enriched Cartesian product. Applying the fibre functor and idempotent completing gives functors:

$$\overline{\overline{H}} \colon \overline{\overline{\mathcal{K} \underset{s}{\boxtimes} \mathcal{L}}} \leftrightarrow \overline{\overline{\mathcal{K}}} \boxtimes \overline{\overline{\mathcal{L}}} \colon \overline{\overline{Z}},$$

with $\overline{\overline{ZH}} = \mathrm{Id}$. We claim that the image of $\overline{\overline{H}}$ is $\overline{\overline{\mathcal{K}}} \underset{G}{\boxtimes} \overline{\overline{\mathcal{L}}}$, the result will then follow. To see this, observe that, when viewing $\overline{\mathcal{Z}(\mathcal{A})}$ as G-graded (super) vector spaces over G, $\overline{\eta}$ is the morphism that takes the degreewise product and includes it into the convolution product. This means that $\overline{H}$ will descend to $\overline{\overline{H}}$ as the functor that takes homogeneous idempotents to their degreewise product, which is what we wanted to show. □

This completes the proof of Theorem 4.27.

4.3.3 From G-crossed braided fusion categories to $\mathcal{Z}(\mathcal{A})$-crossed braided fusion categories

In this section, we will give a construction that produces $\mathcal{Z}(\mathcal{A})$-crossed braided categories from G-crossed braided categories, and then extend this to a symmetric monoidal bifunctor **Fix**. This uses a variation of the G-fixed category construction (see for example [Müg10]).

The G-fixed category

Definition 4.52. Let $\mathcal{C}$ be a (super) G-crossed (or (G,ω)-crossed) braided fusion category. Then the *G-fixed category* $\mathcal{C}^G$ is the $\mathcal{Z}(\mathcal{A})_s$-enriched and tensored category with objects pairs $(c, \{u_g\}_{g\in G})$, where c is an object of $\mathcal{C}$, and the $u_g \colon (c)^g \xrightarrow{\cong} c$ are (even) isomorphisms such that:

$$\begin{tikzcd}
(c)^{gh} \arrow[r, "\cong"] \arrow[d, "u_{gh}"'] & ((c)^h)^g \arrow[d, "(u_h)^g"] \\
c & (c)^g \arrow[l, "u_g"]
\end{tikzcd}$$

commutes for all $g, h \in G$. The hom-objects $\mathcal{C}^G((c,u),(c',u')) \in \mathcal{Z}(\mathcal{A})$ are given by

$$\mathcal{C}^G((c,u),(c',u')) = (\mathcal{C}(c,c'), \mathfrak{b}),$$

where we equip $\mathcal{C}(c,c')$ with the G-action:

$$g\cdot\colon \mathcal{C}(c,c') \xrightarrow{(-)^g} \mathcal{C}((c)^g,(c')^g) \xrightarrow{(u_g)^*\circ(u_{g^{-1}})_*} \mathcal{C}(c,c'),$$

and $\mathfrak{b}$ is the half-braiding defined by, for every $a = (V,\rho) \in \operatorname{Rep}(G)$:

$$\mathfrak{b}\colon \mathcal{C}(c,c')a = \bigoplus_{g\in G} \mathcal{C}(c_g,c'_g)V \xrightarrow{\textbf{Switch}} \bigoplus_{g\in G} V\mathcal{C}(c_g,c'_g) \xrightarrow{\oplus\rho(g)\otimes\mathrm{id}} \bigoplus_{g\in G} V\mathcal{C}(c_g,c'_g),$$

where we have used the direct sum decomposition of $\mathcal{C}$ and used subscript g to denote the homogeneous components, and the switch map is the switch map of (super) vector spaces. Examining the definition (Definition A.40) of the half-braiding in $\mathbf{Vect}_G[G]$, we see that this half-braiding corresponds to taking $\mathcal{C}^G((c,u),(c',u'))$ to be the equivariant vector bundle with fibre over g given by:

$$\mathcal{C}^G((c,u),(c',u'))_g = \mathcal{C}(c_g,c'_g)_0 \oplus \mathcal{C}(c_{\omega g},c'_{\omega g})_1,$$

where the subscripts 0 and 1 denote taking the even and odd summands respectively. Composition is given by the composition of $\mathcal{C}$.

Remark 4.53. The reader might observe that this is a variation of the homotopy fixed point construction for the G-action.

Lemma 4.54. *The G-fixed category is indeed a $\mathcal{Z}(\mathcal{A})_s$-enriched and tensored category.*

Proof. Using Theorems 2.28 or 2.35, we can view $\mathcal{Z}(\mathcal{A})_s$ as the category $\mathbf{Vect}_G[G]$ of G-equivariant vector bundles over G, equipped with the (super) fibrewise tensor product, that we will denote by $\otimes_f$ in both cases.

We need to show that the composition of $\mathcal{C}$ defines a morphism:

$$\mathcal{C}^G((c',u'),(c'',u'')) \otimes_f \mathcal{C}^G((c,u),(c',u')) \to \mathcal{C}^G((c,u),(c'',u'')),$$

that is G-equivariant, factors over the (super) fibrewise tensor product, and is compatible with the specified braiding. For the G-equivariance, we simply observe that $u'_g \circ u'_{g^{-1}} = \mathrm{id}$. To see the composition factors over the fibrewise (super) tensor product, observe that the direct sum decomposition of $\mathcal{C}$ implies that any two morphisms $f\colon c_g \to c'_g$ and $f'\colon c'_h \to c''_h$ will compose to 0 unless $g = h$. For the even part of the hom-objects, this immediately implies that the composition factors through the fibrewise tensor product. To examine what happens for the odd parts of the hom-objects, we will start by assuming that one of the morphisms is odd, say the one between c' and c''. In this case the fibrewise super tensor product computes as (Definition 2.33), using the notation from the proof of Lemma 4.40:

$$\mathcal{C}(c'_{\omega g},c''_{\omega g})_1^g \otimes_f \mathcal{C}(c_{g'},c'_{g'})_0^{g'} = \begin{cases} \left(\mathcal{C}(c'_{\omega g},c''_{\omega g})_1\mathcal{C}(c_{\omega g},c'_{\omega g})_0\right)^g & \text{for } g' = \omega g \\ 0 & \text{otherwise.}\end{cases}$$

We see that this corresponds again to morphisms of different degrees composing to zero. The case where the other morphism is odd is similar. If both are odd, we compute:

$$\mathcal{C}(c'_{\omega g}, c''_{\omega g})_1^g \otimes_f \mathcal{C}(c_{\omega g'}, c'_{\omega g'})_1^{g'} = \begin{cases} \left(\mathcal{C}(c'_{\omega g}, c''_{\omega g})_1 \mathcal{C}(c_{\omega g}, c'_{\omega g})_1\right)^{\omega g} & \text{for } g' = g \\ 0 & \text{otherwise,} \end{cases}$$

from which we again see that the composition factors over the fibrewise super tensor product. The specified braiding is exactly the one $\mathbf{Vect}_G[G]$, so this same observation implies that the composition morphism commutes with the braiding. $\square$

This construction takes G-crossed braided fusion categories to $\mathcal{Z}(\mathcal{A})$-crossed braided fusion categories.

Proposition 4.55. *If $\mathcal{A} = \mathrm{Rep}(G)$ (or $\mathcal{A} = \mathrm{Rep}(G, \omega)$), then $\mathcal{C}^G$ is a $\mathcal{Z}(\mathcal{A})$-crossed braided fusion category, with $\mathcal{Z}(\mathcal{A})$-crossed tensor structure given by:*

$$\begin{aligned} \otimes \colon \mathcal{C}^G \underset{c}{\boxtimes} \mathcal{C}^G &\to \mathcal{C}^G \\ (c, u) \boxtimes (c', u') &\mapsto (cc', u \otimes u'), \end{aligned}$$

and on morphisms by the monoidal structure in $\mathcal{C}$. The $\mathcal{Z}(\mathcal{A})$-crossed braiding is the natural transformation with the same components as the crossed braiding on $\mathcal{C}$.

Proof. The first step is to show the monoidal structure on morphisms really factors over the convolution tensor product. We observe that, as the monoidal structure on $\mathcal{C}$ is graded, we have:

$$(c_1 c_2)_g = \bigoplus_{g_1 g_2 = g} c_{1,g_1} c_{2,g_2}.$$

This gives a decomposition of the hom-object

$$\mathcal{C}(c_1 c_2, c_1 c'_1) = \bigoplus_{g \in G} \bigoplus_{g_1 g_2 = g} \mathcal{C}(c_{1,g_1} c_{2,g_2}, c'_{1,g_1} c'_{2,g_2}).$$

From this, we see that the monoidal structure in $\mathcal{C}$ will indeed factor over the convolution product. To see that the crossed braiding induces a $\mathcal{Z}(\mathcal{A})$-crossed braiding, we observe that the half-braiding on $\mathcal{C}^G(c, c')$ restricts to the G-action on the summands. $\square$

The 2-functor Fix

We now want to extend the G-fixed category construction to functors and natural transformations of (super) G-crossed braided categories.

Definition-Propostion 4.56. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a 1-morphism in G-**XBF** (or (G,ω)-**XBF**). Then we define the *associated G-fixed functor* $\mathbf{Fix}(F)$ as

$$F\colon (c,u) \mapsto (F(c), F(u))$$

on objects and by

$$F_{c,c}\colon \mathcal{C}(c,c') \to \mathcal{C}'(Fc, Fc')$$

on hom-objects. This is a 1-morphism in $\mathcal{Z}(\mathcal{A})$-**XBF**.

Proof. We need to show that this prescription indeed defines a $\mathcal{Z}(\mathcal{A})_s$-enriched functor that is braided monoidal. On objects, there is nothing to show. On hom-objects, we need to show that F acts by morphisms in $\mathcal{Z}(\mathcal{A})$, so is compatible with the prescribed half-braiding, this follows from the G-equivariance of F. The fact that $\mathbf{Fix}(F)$ is braided monoidal is immediate from the definition of the $\mathcal{Z}(\mathcal{A})$-crossed braided monoidal structures on $\mathcal{C}^G$ and $\mathcal{C}^{\prime,G}$. □

To extend **Fix** to 2-morphisms, we define:

Definition 4.57. Let κ be a 2-morphism in G-**XBF** (or (G,ω)-**XBF**) between $F, F'\colon \mathcal{C} \to \mathcal{C}'$. Then $\mathbf{Fix}(\kappa)$ is the $\mathcal{Z}(\mathcal{A})$-enriched natural transformation with components:

$$\mathbf{Fix}(\kappa)_{(c,u)}\colon \mathbb{I}_s \to \mathcal{C}'(F(c), F'(c)),$$

given fibrewise by $\kappa_{c_g}\colon \mathbb{C} \times \{g\} \to \mathcal{C}'(F(c_g), F'(c_g))$.

4.3.4 Equivalence between $\mathcal{Z}(\mathcal{A})$-XBF and G-XBF (or (G,ω)-XBF)

We will now show that the 2-functors $\overline{\overline{\overline{(-)}}}$ and **Fix** are mutually inverse, this will complete the proof of Theorem 4.27:

Proposition 4.58. *The 2-functors $\overline{\overline{\overline{(-)}}}$ and* **Fix** *are mutually inverse.*

As a first step, we will show that $\mathbf{Fix}(\overline{\overline{\overline{\mathcal{K}}}})$ is equivalent to $\overline{\overline{\overline{\mathcal{K}}}}$. To do this, we will need the following two technical lemmas:

Lemma 4.59. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a fully faithful functor on an idempotent complete category $\mathcal{C}$. Then the essential image of F is idempotent complete.*

Proof. Suppose that $f \in \mathrm{End}_{\mathcal{D}}(F(c))$ is an idempotent. By full faithfulness of F, this f is the image of a unique $g \in \mathrm{End}_{\mathcal{C}}(c)$, which is necessarily idempotent. By idempotent completeness of $\mathcal{C}$, there exists an object $b \hookrightarrow c$ corresponding to g, which is mapped to a subobject $F(b) \hookrightarrow F(c)$ corresponding to f under the functor F. □

Lemma 4.60. *Suppose that for each object c in an abelian category $\mathcal{C}$ we have a natural assignment $c \mapsto (i(c)\colon c \to B(c))$, and that for every non-zero c the map $i(c)$ is non-zero. Then $i(c)$ is monic for all c.*

Proof. Suppose that $i(c)$ had some kernel $\mathfrak{k}\colon k \hookrightarrow c$. Then applying our natural assignment to k gives the commutative diagram:

$$\begin{tikzcd} k \arrow[r, "i(k)"] \arrow[d, hook, "\mathfrak{k}"] & B(k) \arrow[d, "B(\mathfrak{k})"] \\ c \arrow[r, "i(c)"] & B(c). \end{tikzcd}$$

The composite along the bottom is zero, as $\mathfrak{k}$ is the kernel of $i(c)$. By naturality, $B(\mathfrak{k})$ is the kernel of $B(i(c))$, and therefore monic. So the bottom composite being zero implies that $i(k)$ is zero, implying that the kernel is trivial. $\square$

We are now in a position to prove that $\mathcal{K}$ and $\mathbf{Fix}(\overline{\overline{\mathcal{K}}})$ are equivalent.

Lemma 4.61. *For each $\mathcal{Z}(\mathcal{A})$-crossed braided fusion category $\mathcal{K}$ there is an equivalence of $\mathcal{Z}(\mathcal{A})$-crossed tensor categories*

$$\mathcal{H}_{\mathcal{K}}\colon \mathcal{K} \to \mathbf{Fix}(\overline{\overline{\mathcal{K}}}),$$

given by taking $k \in \mathcal{K}$ to $(\mathrm{id}_k, \{\mathrm{id}_k\}_{g\in G})$, and on hom-objects by the isomorphism

$$\mathcal{K}(k, k') \cong (\Phi(\overline{\mathcal{K}}), \rho, \mathfrak{b}),$$

where ρ denotes the G-action on $\Phi\mathcal{K}$ coming from the G-action on $\mathcal{K}(k, k')$, and $\mathfrak{b}$ its half-braiding.

Proof. This functor is fully faithful by definition, so we only need to establish essential surjectivity. That is, for every (f, u), we need to give an isomorphism to an object $(\mathrm{id}_k, \{\mathrm{id}_k\}_{g\in G})$. When f is zero, this is trivial, so assume f is non-zero. As the essential image of a fully faithful functor on an idempotent complete category is idempotent complete (Lemma 4.59), it suffices to find a monic morphism

$$(f, u) \to_{\mathbb{I}_s} (\mathrm{id}_k, \{\mathrm{id}_k\}_{g\in G}),$$

this will then correspond to a subobject of $(\mathrm{id}_k, \{\mathrm{id}_k\}_{g\in G})$, which is necessarily in the essential image. If (f, u) has as underlying idempotent $f \in \Phi\overline{\mathcal{K}}(k', k')$, we will produce a morphism to

$$\mathbb{C}[G]^* \cdot (\mathrm{id}_{k'}, \{\mathrm{id}_{k'}\}_{g\in G}) = (\mathrm{id}_k, \{\mathrm{id}_k\}_{g\in G}),$$

where $k = \mathbb{C}[G]^* k'$ and we equip $\mathbb{C}[G]$ with the left action of G. To produce this morphism, observe that f defines a morphism in $\overline{\overline{\mathcal{K}}}(f, \mathrm{id}_{k'})$, and therefore gives rise to a morphism in $\mathbf{Fix}(\overline{\overline{\mathcal{K}}})((f, u), (\mathrm{id}_{k'}, \{\mathrm{id}_{k'}\}_{g\in G}))$. The image under the G-action for $g \in G$ (see Definition 4.52) of f is:

$$f \overset{(-)^g}{\longmapsto} f^g \overset{-\circ (u_g)^{-1}}{\longmapsto} f^g u_g^{-1}.$$

By adjunction in $\mathcal{Z}(\mathcal{A})_s$, a morphism degree $\mathbf{I}_s$ to $(\mathrm{id}_k, \{\mathrm{id}_k\}_{g\in G})$ is the same as a fibrewise map:

$$\tilde{f}\colon \mathbb{C}[G] \otimes_s \mathbb{I}_s \to \mathbf{Fix}(\overline{\overline{\mathcal{K}}})((f,u),(\mathrm{id}_{k'}, \{\mathrm{id}_{k'}\}_{g\in G}))$$

that is equivariant for the G action. The G-equivariant vector bundle $\mathbb{C}[G]\otimes_s \mathbb{I}_s$ is the bundle $\mathbb{C}[G] \times G$, where G acts on the fibres by left multiplication. We define $\tilde{f}$ by $f|_{\{g\}\times G} = f^g u_g^{-1}$. To show that the morphism

$$\mathbb{I}_s \to \mathbf{Fix}(\overline{\overline{\mathcal{K}}})((f,u),(\mathrm{id}_k, \{\mathrm{id}_k\}_{g\in G}))$$

is obtained in this way is monic, by Lemma 4.60 it suffices to show that $\tilde{f}$ is non-zero. Restricting $\tilde{f}$ to $\{e\} \times G$ gives f, which is assumed to be non-zero. Therefore, we have produced a monic morphism from (f,u) to $(\mathrm{id}_k, \{\mathrm{id}_k\}_{g\in G})$, which by Lemma 4.59 corresponds to an subobject of the form $(\mathrm{id}_l, \{\mathrm{id}_l\}_{g\in G})$ for some $l \in \mathcal{K}$.

It is clear from the definition of $\mathcal{H}_{\mathcal{K}}$ that it will be a functor of $\mathcal{Z}(\mathcal{A})$-crossed braided categories. □

As a second step, we will show that the $\mathcal{H}_{\mathcal{K}}$ are natural in the sense that:

Lemma 4.62. *Let $F\colon \mathcal{K} \to \mathcal{K}'$ be a 1-morphism in $\mathcal{Z}(\mathcal{A})$-**XBF**. Then F and the image of F under $\mathbf{Fix} \circ \overline{\overline{(-)}}$ fit into a commutative diagram:*

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "\mathcal{H}_{\mathcal{K}}"] \arrow[d, "F"] & \mathbf{Fix}(\overline{\overline{\mathcal{K}}}) \arrow[d, "\mathbf{Fix}(\overline{\overline{F}})"] \\
\mathcal{K}' \arrow[r, "\mathcal{H}_{\mathcal{K}'}"] & \mathbf{Fix}(\overline{\overline{\mathcal{K}'}}).
\end{tikzcd}$$

Proof. Let $k \in \mathcal{K}$. Under the top composite in the diagram, this object is sent to

$$(F(\mathrm{id}_k), F(\{\mathrm{id}_k\}_{g\in G}) = (\mathrm{id}_{F(k)}, \{\mathrm{id}_{F(k)}\}_{g\in G}),$$

and the bottom composite is the same. On morphisms, it is similarly clear that the diagram commutes on hom-objects. □

The two Lemmas 4.61 and 4.62 together imply that $\mathbf{Fix}\circ\overline{\overline{(-)}}$ is isomorphic to the identity on $\mathcal{Z}(\mathcal{A})$-**XBF**. For the composite the other way around, we first prove:

Lemma 4.63. *Let $\mathcal{C}$ be a (super)-G-crossed braided fusion category. Then the categories $\overline{\overline{\mathbf{Fix}(\mathcal{C})}}$ and $\mathcal{C}$ are equivalent.*

Proof. We will define a dominant fully faithful functor

$$\hat{\mathcal{P}}_{\mathcal{C}}\colon \Phi\overline{\mathbf{Fix}(\mathcal{C})} \to \mathcal{C},$$

after idempotent completion this will descent to an equivalence of categories. On objects, this functor is given by $(c, u) \mapsto c$, on morphisms we use the isomorphism

$$\Phi\overline{\mathbf{Fix}}((c,u),(c',u')) \cong \mathcal{C}(c,c'),$$

as $\Phi\overline{(-)}$ simply forgets the G-action and half-braiding. This functor is clearly fully faithful. To see that it is dominant, observe that for any object $c \in \mathcal{C}$, the object $\oplus_{g\in G} c^g$ is a fixed point for the G-action, and therefore $(\oplus_{g\in G} c^g, \{\mathrm{id}\}_{g\in G})$ defines an object in $\mathbf{Fix}(\mathcal{C})$, which our functor will send to $\oplus_{g\in G} c^g$. This object has c as a summand, so we see our functor is indeed dominant. $\square$

Lemma 4.64. *Denote the equivalence from Lemma 4.63 by*

$$\mathcal{P}_{\mathcal{C}} \colon \overline{\overline{\mathbf{Fix}(\mathcal{C})}} \to \mathcal{C}.$$

Then, for any 1-morphism F in G-$\mathbf{XBF}$ (or (G,ω)-$\mathbf{XBF}$), the diagram

$$\begin{tikzcd} \overline{\overline{\mathbf{Fix}(\mathcal{C})}} \arrow[r, "\mathcal{P}_{\mathcal{C}}"] \arrow[d, "\overline{\overline{\mathbf{Fix}(F)}}"] & \mathcal{C} \arrow[d, "F"] \\ \overline{\overline{\mathbf{Fix}(\mathcal{C}')}} \arrow[r, "\mathcal{P}_{\mathcal{C}'}"] & \mathcal{C}' \end{tikzcd}$$

commmutes.

Proof. It suffices to show that the diagram

$$\begin{tikzcd} \Phi\overline{\mathbf{Fix}(\mathcal{C})} \arrow[r, "\hat{\mathcal{P}}_{\mathcal{C}}"] \arrow[d, "\Phi\overline{\mathbf{Fix}(F)}"'] & \mathcal{C} \arrow[d, "F"] \\ \Phi\overline{\mathbf{Fix}(\mathcal{C}')} \arrow[r, "\hat{\mathcal{P}}_{\mathcal{C}'}"] & \mathcal{C}' \end{tikzcd}$$

commutes, as it will descent to the desired diagram after idempotent completion. On an object $(c, u) \in \Phi\overline{\mathbf{Fix}(\mathcal{C})}$, the bottom route becomes

$$(c,u) \mapsto (Fc, Fu) \mapsto Fc,$$

which agrees with the top route. A similar diagram chase shows that this diagram commutes on hom-objects. $\square$

The Lemmas 4.63 and 4.64 together show that the composite $\overline{\overline{\mathbf{Fix}(-)}}$ is naturally isomorphic to the identity on G-$\mathbf{XBF}$ (or (G,ω)-$\mathbf{XBF}$).

This finishes the proof of Proposition 4.58, and with that, the proof of Theorem 4.27.

Chapter 5

From Braided Fusion Categories over a Symmetric Fusion Category $\mathcal{A}$ to $\mathcal{Z}(\mathcal{A})$-Crossed Braided Categories

In this chapter we discuss how to obtain a $\mathcal{Z}(\mathcal{A})$-crossed braided fusion category from a braided fusion category containing $\mathcal{A}$. This is the subject of the first section, Section 5.1, and Theorem 5.17. We then proceed to show that this construction has an inverse, this is Theorem 5.41.

5.1 Enriching

Our enriching procedure to obtain from a braided fusion category $\mathcal{C}$ containing a symmetric fusion category $\mathcal{A}$ will be divided into two steps. The first is enriching $\mathcal{C}$ over $\mathcal{A}$, we will denote the result by $\underleftarrow{\mathcal{C}}$. It turns out $\underleftarrow{\mathcal{C}}$ is $\mathcal{A}$-tensor (Definition A.20), but not braided. We will examine the failure of this category to be braided in some detail, this will motivate the next step in the construction. In the second step of the construction, we boost our $\mathcal{A}$-enrichment to a $\mathcal{Z}(\mathcal{A})_s$-enrichment, taking care to define the half-braidings to ensure the result is $\mathcal{Z}(\mathcal{A})$-crossed braided (Definition 4.20). We will show the result is indeed $\mathcal{Z}(\mathcal{A})$-braided in Theorem 5.17.

5.1.1 Enriching over a symmetric subcategory

The enriched category

In this section, we consider the situation where we have a braided fusion category $\mathcal{C}$ and containing a symmetric fusion category $\mathcal{A}$.

Definition 5.1. Let $\mathcal{C}$ be a fusion category containing a symmetric fusion category $\mathcal{A}$. The *left-associated $\mathcal{A}$-enriched category* $\underleftarrow{\mathcal{C}}$ has the same objects as $\mathcal{C}$ and $\underleftarrow{\mathcal{C}}(c, c')$ is defined by

$$\mathcal{A}(a, \underleftarrow{\mathcal{C}}(c, c')) = \mathcal{C}(ac, c'). \tag{5.1}$$

Here the action of $\mathcal{A}$ comes from the postcomposing $\mathcal{A} \to \mathcal{Z}(\mathcal{C})$ with the forgetful functor to $\mathcal{C}$. The composition morphisms,

$$\circ \colon \underleftarrow{\mathcal{C}}(c', c'') \otimes \underleftarrow{\mathcal{C}}(c, c') \to \underleftarrow{\mathcal{C}}(c, c''),$$

are defined by observing that we have the following string of canonical isomorphisms:

$$\begin{aligned} \mathcal{A}(a, \underleftarrow{\mathcal{C}}(c', c'') \otimes \underleftarrow{\mathcal{C}}(c, c')) &\cong \mathcal{A}(\underleftarrow{\mathcal{C}}(c', c'')^* \otimes a, \underleftarrow{\mathcal{C}}(c, c')) \\ &\cong \mathcal{C}(\underleftarrow{\mathcal{C}}(c', c'')^* \otimes ac, c') \\ &\cong \mathcal{C}(ac, \underleftarrow{\mathcal{C}}(c', c'') \otimes c') \\ &\xrightarrow{\mathrm{ev}} \mathcal{C}(ac, c'') \\ &\cong \mathcal{A}(a, \underleftarrow{\mathcal{C}}(c, c'')). \end{aligned} \tag{5.2}$$

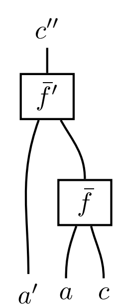
Here ev is the unit of the adjunction given by (5.1), c.f. Definition A.11.

Similarly, we define the *right-associated $\mathcal{A}$-enriched category* $\underrightarrow{\mathcal{C}}$, by representing $a \mapsto \mathcal{C}(ca, c')$.

Observe that $\mathrm{Hom}_{\underleftarrow{\mathcal{C}}}(c, c') = \mathcal{C}(c, c')$. This means that we can view the mate $\bar{f}$ (Definition A.4) of $f \colon c \to_a c'$ as a morphism in $\mathcal{C}$. In terms of mates and the composition in $\mathcal{C}$, the composition of $f \colon c \to_a c'$ and $f' \colon c' \to_{a'} c''$ in $\underleftarrow{\mathcal{C}}$ is given by

$$f' \circ f = \underline{\bar{f}'(\mathrm{id}_{a'} \otimes \bar{f})}, \tag{5.3}$$

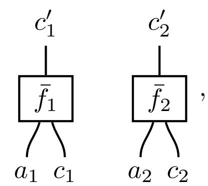
which in string diagrams reads as:



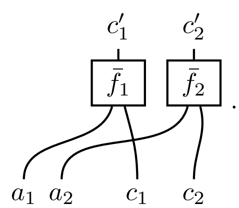
. (5.4)

Remark 5.2. Both $\underleftarrow{\mathcal{C}}$ and $\underrightarrow{\mathcal{C}}$ are tensored $\underleftarrow{\mathcal{C}}$ over $\mathcal{A}$. For $\underrightarrow{\mathcal{C}}$, the tensoring induces a functor $\mathcal{A}^{\mathrm{mop}} \to \mathrm{End}(\underrightarrow{\mathcal{C}})$, where $\mathcal{A}^{\mathrm{mop}}$ denotes the monoidal opposite of $\mathcal{A}$.

The $\mathcal{A}$-product $\underset{\mathcal{A}}{\boxtimes}$ (Definition A.17) between $\mathcal{A}$-enriched categories obtained in this way has some nice features. Corresponding to the product of $f_1 \colon c_1 \to_{a_1} c_1'$ and $f_2 \colon c_2 \to_{a_2} c_2'$ there is, by using the tensor product of the tensor structures a map $\bar{f}_1 \otimes_{\mathcal{C}} \bar{f}_2$. It is tempting to represent this in string diagrams as:



Care should be taken, however, that, by Equation (A.11), the position of the a's is immaterial. To avoid confusion, we will therefore always keep the objects of $\mathcal{A}$ to the left when we are dealing with left enrichments. In drawing string diagrams, this does mean that we need to cross $\mathcal{A}$-strands past $\mathcal{C}$-strands. To emphasise such crossings are not actual braidings in $\mathcal{C}$, we will draw them unresolved as follows:



(5.5)

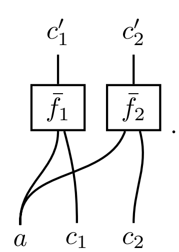
When considering a morphism $f \colon c_1 \boxtimes c_2 \to_a c_1' \boxtimes c_2'$, we will give a string diagram presentation by first picking a factorisation (t, f_1, f_2):

$$f \colon a \xrightarrow{t} a_1 a_2 \xrightarrow{f_1 f_2} \underleftarrow{\mathcal{C}}(c_1, c_1') \underleftarrow{\mathcal{C}}(c_2, c_2'), \tag{5.6}$$

and then using the tensor isomorphism to find mates for f_1 and f_2. There are many different choices of factorisations for a given f. In terms of the triples, we have the equivalence relation

$$(t, f_1 \circ g_1, f_2 \circ g_2) \sim (g_1 g_2 \circ t, f_1, f_2).$$

A factorisation (t, f_1, f_2) can be presented in string diagrams by:



(5.7)

Here the trivalent vertex represents the morphism $t\colon a \to a_1a_2$ from Equation (5.6).

Enriched monoidal structure

Definition 5.3. The tensor structure on $\mathcal{C}$ induces an associated $\mathcal{A}$-monoidal structure on $\underleftarrow{\mathcal{C}}$ (and similarly on $\underrightarrow{\mathcal{C}}$). This $\mathcal{A}$-monoidal structure is defined as follows. The functor $\otimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ induces map on objects $\underleftarrow{\mathcal{C}} \boxtimes_{\mathcal{A}} \underleftarrow{\mathcal{C}} \to \underleftarrow{\mathcal{C}}$, where we do still need to specify what it does on the additional morphisms. We should therefore construct a map

$$\underset{\underleftarrow{\mathcal{C}}}{\otimes}\colon \underleftarrow{\mathcal{C}}(c_1,c_1') \underset{\mathcal{A}}{\otimes} \underleftarrow{\mathcal{C}}(c_2,c_2') \to \underleftarrow{\mathcal{C}}(c_1c_2,c_1'c_2'). \tag{5.8}$$

To do this, consider the following composite:

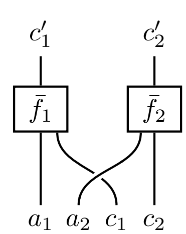
$$\begin{aligned}\mathcal{A}(a_1,\underleftarrow{\mathcal{C}}(c_1,c_1')) \underset{\mathbf{Vect}}{\otimes} \mathcal{A}(a_2,\underleftarrow{\mathcal{C}}(c_2,c_2')) &= \mathcal{C}(a_1c_1,c_1') \underset{\mathbf{Vect}}{\otimes} \mathcal{C}(a_2c_2,c_2')\\ &\xrightarrow{\otimes_{\mathcal{C}}} \mathcal{C}(a_1c_1a_2c_2,c_1'c_2')\\ &\xrightarrow{(\beta_{a_2,c_1})^*} \mathcal{C}(a_1a_2c_1c_2,c_1'c_2')\end{aligned} \tag{5.9}$$

with the monoidal structure in $\mathcal{C}$ in the second line and the braiding between a_2 and c_1 in the last line. Setting $a_i = \underleftarrow{\mathcal{C}}(c_i,c_i')$ for $i=1,2$, we obtain our desired map from Equation (5.8) as the image of the tensor product of the identities under this morphism.

In terms of mates, this translates to the following. Let $f_1\colon c_1 \to_{a_1} c_1'$ and $f_2\colon c_2 \to_{a_2} c_2'$, following the above recipe we find:

$$f_1 \underset{\underleftarrow{\mathcal{C}}}{\otimes} f_2 = \underline{\bar{f}_1 \underset{\mathcal{C}}{\otimes} \bar{f}_2(\mathrm{id}_{a_1} \underset{\mathcal{C}}{\otimes} \beta_{a_2,c_1} \underset{\mathcal{C}}{\otimes} \mathrm{id}_{c_2})}. \tag{5.10}$$

In string diagrams, this becomes:



. (5.11)

Remark 5.4. To make this definition, it would have sufficed to assume that $\mathcal{A}$ comes equipped with a central functor $\mathcal{A}\to\mathcal{Z}(\mathcal{C})\to\mathcal{C}$.

Lemma 5.5. *The categories $\underleftarrow{\mathcal{C}}$ and $\underrightarrow{\mathcal{C}}$ are $\mathcal{A}$-monoidal, with the monoidal structure from Definition 5.3.*

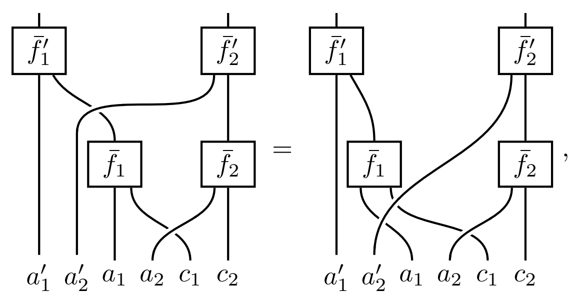
Proof. We will only provide a proof for $\underleftarrow{\mathcal{C}}$, the case of $\underrightarrow{\mathcal{C}}$ is similar. We need to prove the structure above satisfies the interchange law, i.e. that the proposed $\mathcal{A}$-monoidal structure is indeed a functor. Checking functoriality boils down to checking that the following diagram commutes:

$$\begin{tikzcd}
\underleftarrow{\mathcal{C}}(c_1 \boxtimes c_2, c_1' \boxtimes c_2') \otimes \underleftarrow{\mathcal{C}}(c_1' \boxtimes c_2', c_1'' \boxtimes c_2'') \arrow[r, "\circ_{(\underleftarrow{\mathcal{C}} \boxtimes_{\mathcal{A}} \underleftarrow{\mathcal{C}})}"] \arrow[d, "\otimes_{\underleftarrow{\mathcal{C}}} \boxtimes_{\mathcal{A}} \otimes_{\underleftarrow{\mathcal{C}}}"] & \underleftarrow{\mathcal{C}}(c_1, c_1'') \otimes \underleftarrow{\mathcal{C}}(c_2, c_2'') \arrow[d, "\otimes_{\underleftarrow{\mathcal{C}}}"] \\
\underleftarrow{\mathcal{C}}(c_1c_2, c_1'c_2') \otimes \underleftarrow{\mathcal{C}}(c_1'c_2', c_1''c_2'') \arrow[r, "\circ_{\underleftarrow{\mathcal{C}}}"] & \underleftarrow{\mathcal{C}}(c_1c_2, c_1''c_2'').
\end{tikzcd}$$

We will do this by checking that the precomposition of the two routes in this diagram with

$$f_1 \otimes f_2 \otimes f_1' \otimes f_2' \colon a_1a_2a_1'a_2' \to \underleftarrow{\mathcal{C}}(c_1, c_1')\underleftarrow{\mathcal{C}}(c_2, c_2')\underleftarrow{\mathcal{C}}(c_1', c_1'')\underleftarrow{\mathcal{C}}(c_2', c_2'')$$

are the same. This will be the case if and only if their mates are equal. Using Equations (5.4) and (5.11) we see that we need to check:



(5.12)

and this equation holds by naturality of the braiding in $\mathcal{C}$.

The associators in $\mathcal{C}$ will descend to morphisms in $\underleftarrow{\mathcal{C}}$ and still satisfy the pentagon equations. We have to convince ourselves that these morphisms define a natural isomorphism, with respect to the extra morphisms in the enriched hom-objects $\underleftarrow{\mathcal{C}}(c, c')$ for $c, c' \in \mathcal{C}$. But by (A.2), all these extra morphisms are just morphisms $ac \to c'$ for some $a \in \mathcal{A}$. Using the pentagon equations on these morphisms, this means the associators from $\mathcal{C}$ will also be natural for these extra morphisms. □

5.1.2 Braiding

In the previous section, we only used the half-twists $\beta_{a,c}$ for $a \in \mathcal{A}$ and $c \in \mathcal{C}$, and the braiding in $\mathcal{A}$. From here onward, we will need that $\mathcal{C}$ is itself braided, and that $\mathcal{A}$ is a full symmetric subcategory of $\mathcal{A}$.

A second $\mathcal{A}$-tensor structure

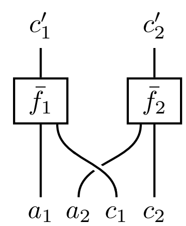
Since we made a choice to use β rather than β^{-1} in Definition 5.3, we also have:

Definition 5.6. We define $-\otimes^{\beta}-\colon \underleftarrow{\mathcal{C}} \boxtimes_{\mathcal{A}} \underleftarrow{\mathcal{C}} \to \underleftarrow{\mathcal{C}}$, by taking it to be $-\otimes_{\mathcal{C}}-$ on objects and on morphisms the image of the identity under the composite

$$\begin{aligned}\mathcal{A}(a_1, \underleftarrow{\mathcal{C}}(c_1, c_1')) \otimes \mathcal{A}(a_2, \underleftarrow{\mathcal{C}}(c_2, c_2')) = \mathcal{C}(a_1c_1, c_1') &\underset{\mathbf{Vect}}{\otimes} \mathcal{C}(a_2c_2, c_2') \\ &\xrightarrow{\otimes_{\mathcal{C}}} \mathcal{C}(a_1c_1a_2c_2, c_1'c_2') \\ &\xrightarrow{(\beta^{-1}_{c_1,a_2})^*} \mathcal{C}(a_1a_2c_1c_2, c_1'c_2'),\end{aligned}$$

where $a_i = \underleftarrow{\mathcal{C}}(c_i, c_i')$ for $i = 1, 2$.

The proof that this indeed specifies an $\mathcal{A}$-monoidal structure is analogous to the proof of Lemma 5.5. In string diagrams for the mates of $f_1\colon c_1 \to_{a_1} c_1'$, and $f_2\colon c_2 \to_{a_2} c_2'$ this monoidal structure gives:



. (5.13)

A Problem with the Braiding

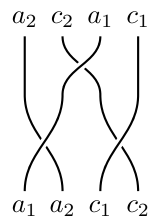
Remark 5.7. One can attempt to lift the braiding of $\mathcal{C}$ to a braiding on $\underleftarrow{\mathcal{C}}$. A problem one encounters here is that the braiding will no longer be natural with respect to the additional morphisms. In particular, the diagram

$$\begin{tikzcd} c_1c_2 \arrow[r, "\beta_{c_1,c_2}"] \arrow[d, "f_1\otimes f_2", "a_1a_2"'] & c_2c_1 \arrow[d, "f_2\otimes f_1", "a_2a_1"'] \\ c_1'c_2' \arrow[r, "\beta_{c_1',c_2'}"'] & c_2'c_1' \end{tikzcd}$$

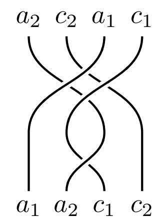
fails to commute in general. It is interesting to examine its failure to commute. In terms of the mates, this diagram becomes the outside of:

$$\begin{tikzcd} a_1a_2c_1c_2 \arrow[r, "\beta_{a_1,a_2}\otimes\beta_{c_1,c_2}"] \arrow[d, "\beta_{a_2,c_1}"] & a_2a_1c_2c_1 \arrow[d, "\beta_{a_1,c_2}"] \\ a_1c_1a_2c_2 \arrow[r, "\beta_{a_1c_1,a_2c_2}"] \arrow[d, "\bar{f}_1\otimes\bar{f}_2"] & a_2c_2a_1c_1 \arrow[d, "\bar{f}_2\otimes\bar{f}_1"] \\ c_1'c_2' \arrow[r, "\beta_{c_1',c_2'}"'] & c_2'c_1' \end{tikzcd},$$

here the braiding β_{a_1,a_2} in the top row comes from the switch map for the $\mathcal{A}$-product that was implicit in the previous diagram. The map in the middle will help us understand the failure of commutativity. Note that, by naturality of the braiding in $\mathcal{C}$, the lower square of the diagram does commute. It therefore suffices to consider the top square, in string diagrams the top and bottom routes read



and



, (5.14)

respectively. We see that these diagrams differ from each other by a precomposition with the braiding monodromy $\beta_{c_1,a_2}\beta_{a_2,c_1}$ between a_2 and c_1.

5.1.3 To $\mathcal{Z}(\mathcal{A})$-Crossed Braided

One of the desiderata for the reduced tensor product is that it tensors objects with the same braiding behaviour with respect to $\mathcal{A}$ to objects with that same behaviour. We can do this by doing a construction at the level of the Hom-objects. In particular, we will make use of a symmetric tensor product $\otimes$ on $\mathcal{Z}(\mathcal{A})$ that picks out the parts of objects where their braidings with $\mathcal{A}$ agree (see Theorem 2.22). The definition of $\mathcal{Z}(\mathcal{A})$ is given in Definition A.36, and this symmetric tensor product is introduced in Section 2.3. In the language of Chapter 4, we want to prove:

Theorem 5.8. *Let $\mathcal{C}$ be a braided fusion category containing a full symmetric subcategory $\mathcal{A}$, then the category $\underleftarrow{\mathcal{C}}$ defined in Definitions 5.10, 5.11, 5.13, with $\mathcal{Z}(\mathcal{A})_s$-tensoring from Proposition 5.16, crossed tensor structure from Proposition 5.17 and crossed braiding from Proposition 5.22 is $\mathcal{Z}(\mathcal{A})$-crossed braided fusion (Definitions 4.16 and 4.20).*

The $\mathcal{Z}(\mathcal{A})_s$-enrichment

We will now show that the $\mathcal{A}$-enrichment from the previous sections gives rise to an enrichment over $(\mathcal{Z}(\mathcal{A}), \otimes_s)$. That is, we need to define the enriched hom-functor with values in $\mathcal{Z}(\mathcal{A})$, the composition, and the identity morphisms.

Proposition 5.9. *Let $\mathcal{C}$ be a braided tensor category containing a spherical symmetric fusion category $\mathcal{A}$. Then the category $\underleftarrow{\mathcal{C}}$ defined above is a $\mathcal{Z}(\mathcal{A})_s$-enriched and tensored category.*

The first step towards enriching $\mathcal{C}$ over $\mathcal{Z}(\mathcal{A})_s$ is:

Definition-Propostion 5.10. Let $\mathcal{C}$ be a braided category containing $\mathcal{A}$ as a braided subcategory. Then the functor $\underleftarrow{\mathcal{C}} : \underleftarrow{\mathcal{C}}^{\mathrm{op}} \times \underleftarrow{\mathcal{C}} \to \mathcal{A}$ lifts (with respect to the forgetful functor) to a functor $\underleftarrow{\mathcal{C}}$ to $\mathcal{Z}(\mathcal{A})$. The half-braiding $\mathfrak{b}$ on

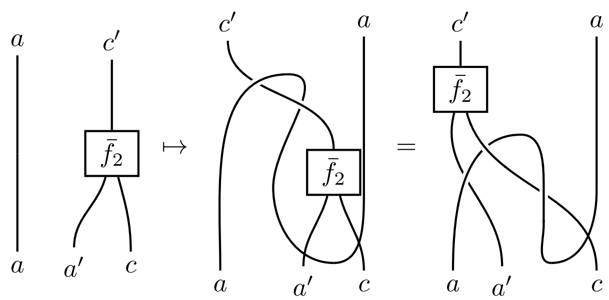
$$\underleftarrow{\mathcal{C}}(c, c') = (\underleftarrow{\mathcal{C}}(c, c'), \mathfrak{b})$$

is defined by:

$$a\underleftarrow{\mathcal{C}}(c,c') \xrightarrow{\cong} \underleftarrow{\mathcal{C}}(c,ac') \xrightarrow{(\beta^{-1}_{a,c'}\beta^{-1}_{c',a})_*} \underleftarrow{\mathcal{C}}(c,ac') \xrightarrow{\cong} a\underleftarrow{\mathcal{C}}(c,c') \xrightarrow{s} \underleftarrow{\mathcal{C}}(c,c')a, \quad (5.15)$$

where s denotes the symmetry in $\mathcal{A}$. The image under $\underleftarrow{\mathcal{C}}$ of (c,c') will be called the $\mathcal{Z}(\mathcal{A})$-*enriched hom-object.*

Proof. Using that $a = \underleftarrow{\mathcal{C}}(\mathbb{I},a)$, we can unpack the half-braiding from Equation (5.15) in terms of the mates for $\underline{\mathrm{id}_a}\colon \mathbb{I} \to_a a$ and $f_2\colon c \to_{a'} c'$ as:



(5.16)

where we interpret the last diagram as the mate to the tensor product of a morphism $aa'a^* \to \underleftarrow{\mathcal{C}}(c,c')$ and $\underline{\mathrm{id}_a}\colon \mathbb{I} \to a$. To show the half-braiding satisfies (A.14), one uses that the braiding in $\mathcal{C}$ satisfies (A.14).

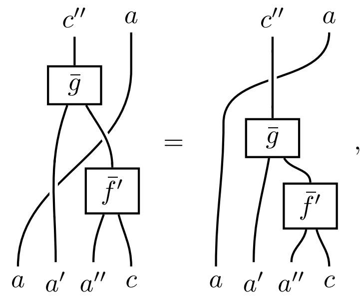
What we have shown so far, is that every hom-object can be viewed as an object in the Drinfeld centre of $\mathcal{A}$. To show this defines a functor from $\underleftarrow{\mathcal{C}}^{\mathrm{op}} \times \underleftarrow{\mathcal{C}}$ to $\mathcal{Z}(\mathcal{A})$, we also need to establish that this is compatible with the morphisms. To do this, we need to check that the morphisms f^* and g_* induced by morphisms $f\colon b \to_{a'''} c$ and $g\colon c \to_{a'} c'$ are morphisms in $\mathcal{Z}(\mathcal{A})$. That is, we need to show that

$$\begin{tikzcd}
a\underleftarrow{\mathcal{C}}(c,c') \arrow[r, "\beta_{\underleftarrow{\mathcal{C}}(c,c')}"] \arrow[d, "\mathrm{id}\otimes f^*"] & \underleftarrow{\mathcal{C}}(c,c')a \arrow[d, "f^*\otimes\mathrm{id}"] \\
a\underleftarrow{\mathcal{C}}(b,c') \arrow[r, "\beta_{\underleftarrow{\mathcal{C}}(b,c')}"] & \underleftarrow{\mathcal{C}}(b,c')a
\end{tikzcd}$$

and

$$\begin{tikzcd}
a\underleftarrow{\mathcal{C}}(c,c') \arrow[r, "\beta_{\underleftarrow{\mathcal{C}}(c,c')}"] \arrow[d, "\mathrm{id}\otimes g_*"] & \underleftarrow{\mathcal{C}}(c,c')a \arrow[d, "g_*\otimes\mathrm{id}"] \\
a\underleftarrow{\mathcal{C}}(c,c'') \arrow[r, "\beta_{\underleftarrow{\mathcal{C}}(c,c'')}"] & \underleftarrow{\mathcal{C}}(b,c'')a
\end{tikzcd}$$

commute. The former is obvious, the latter comes down to checking, for each $f' \colon c \to_{a''} c'$, that



(5.17)

which follows from the naturality of the braiding and the fact that a and a' are transparent to each other. ☐

To show that the $\mathcal{A}$-enrichment really can be modified to an $\mathcal{Z}(\mathcal{A})_s$-enrichment, we also need to establish that there is a composition morphism. This composition morphism will factor over the symmetric tensor product $\otimes_s$ for $\mathcal{Z}(\mathcal{A})$, see Definition 2.11. We will denote the resulting category by $\underleftarrow{\mathcal{C}}$.

Definition 5.11. Let $c, c', c'' \in \mathcal{C}$, and let $\underleftarrow{\mathcal{C}}(c, c') = (\underleftarrow{\mathcal{C}}(c, c'), \beta')$ and $\underleftarrow{\mathcal{C}}(c', c'') = (\underleftarrow{\mathcal{C}}(c', c''), \beta'')$ denote the lifts of the $\mathcal{A}$-enriched hom to $\mathcal{Z}(\mathcal{A})$ from Definition 5.10. Then we define the *composition morphism for the* $\mathcal{Z}(\mathcal{A})_s$*-enrichment* by the composite:

$$\Phi(\underleftarrow{\mathcal{C}}(c', c'') \otimes_s \underleftarrow{\mathcal{C}}(c, c')) \hookrightarrow \underleftarrow{\mathcal{C}}(c', c'')\underleftarrow{\mathcal{C}}(c, c') \xrightarrow{\circ} \underleftarrow{\mathcal{C}}(c, c''), \tag{5.18}$$

on the underlying objects in $\mathcal{A}$.

In order for this definition to make sense, we need the composite from Equation (5.18) to define a morphism in $\mathcal{Z}(\mathcal{A})$:

Lemma 5.12. *The composite from Equation* (5.18) *is a morphism in* $\mathcal{Z}(\mathcal{A})$.

Proof. We need to check that the morphism commutes with the braiding. That is, we need to show that the diagram

$$\begin{tikzcd}
a(\underleftarrow{\mathcal{C}}(c', c'') \otimes_s \underleftarrow{\mathcal{C}}(c, c')) \arrow[r, hook] \arrow[d, "\beta"] & a\underleftarrow{\mathcal{C}}(c', c'')\underleftarrow{\mathcal{C}}(c, c') \arrow[r, "\circ"] \arrow[d, "(\mathrm{id}\otimes s)\circ((\beta^{-1}_{a,c''}\circ\beta^{-1}_{c'',a})_*\otimes\mathrm{id})"'] & a\underleftarrow{\mathcal{C}}(c, c'') \arrow[d, "(\beta^{-1}_{a,c''}\circ\beta^{-1}_{c'',a})_*"] \\
(\underleftarrow{\mathcal{C}}(c', c'') \otimes_s \underleftarrow{\mathcal{C}}(c, c'))a \arrow[r, hook] & \underleftarrow{\mathcal{C}}(c', c'')\underleftarrow{\mathcal{C}}(c, c')a \arrow[r, "\circ"] & \underleftarrow{\mathcal{C}}(c, c'')a
\end{tikzcd}$$

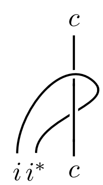
commutes. Here, leftmost square is the definition of the half-braiding (Equation 2.12) on $\underleftarrow{\mathcal{C}}(c', c'')\otimes_s \underleftarrow{\mathcal{C}}(c, c')$ in terms of the half-braiding on $\underleftarrow{\mathcal{C}}(c', c'')$ (Equation 5.15), where we have used monoidality of the symmetry in $\mathcal{A}$ to compose two instances of the symmetry. In the rightmost square, both routes are using the composition morphism and then composing with the braiding in $\mathcal{C}$, so this will commute by the naturality of the braiding. ☐

Additionally, we need to give for each $c \in \mathcal{C}$ an identity morphism $1_c \colon \mathbb{I}_s \to \underleftarrow{\mathcal{C}}(c,c)$ and check that it indeed specifies an identity in the sense that the following diagram commutes

$$\begin{tikzcd}
\underleftarrow{\mathcal{C}}(c,c') \otimes_s \mathbb{I}_s \arrow[r, "\rho"] \arrow[d, "\mathrm{id} \otimes_s 1_c"'] & \underleftarrow{\mathcal{C}}(c',c) \\
\underleftarrow{\mathcal{C}}(c,c') \otimes_s \underleftarrow{\mathcal{C}}(c,c) \arrow[ur, "\circ"] &
\end{tikzcd} , \tag{5.19}$$

where ρ denotes the left unitor, as well as the corresponding diagram for the left unitor.

Definition 5.13. The $\mathcal{Z}(\mathcal{A})$-*unit morphism* $1_c \colon \mathbb{I}_s \to \underleftarrow{\mathcal{C}}(c,c)$ for $c \in \mathcal{C}$ is the mate for the morphism

$$\sum_{i \in \mathcal{O}(\mathcal{A})}$$ 

.

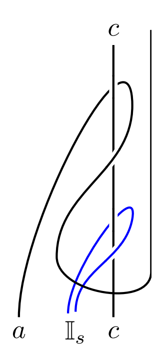
We need to check that this indeed specifies a morphism in $\mathcal{Z}(\mathcal{A})$, and that it satisfies (5.19).

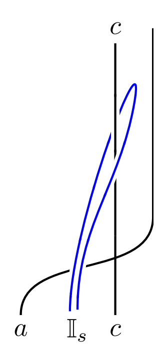
Lemma 5.14. *The unit morphism is a morphism in $\mathcal{Z}(\mathcal{A})$, that is:*

$$\begin{tikzcd}
a\mathbb{I}_s \arrow[r, "1_c"] \arrow[d, "\beta_{a,\mathbb{I}_s}"] & a\underleftarrow{\mathcal{C}}(c,c) \arrow[d, "\beta_{a,\underleftarrow{\mathcal{C}}(c,c)}"] \\
\mathbb{I}_s a \arrow[r, "1_c"] & \underleftarrow{\mathcal{C}}(c,c)a,
\end{tikzcd}$$

commutes.

Proof. Recalling that the braiding $\beta_{a,\underleftarrow{\mathcal{C}}(c,c)}$ was computed in terms of mates in Equation (5.16), the top and bottom routes compute as



and


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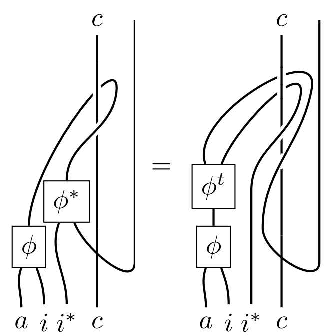
respectively. The latter has summands

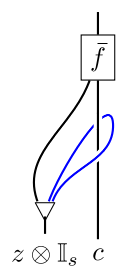


,

where the ϕ give a resolution of the identity on ai. The last diagram sums to the top route, remembering that the objects in $\mathcal{A}$ are transparent to each other. $\square$

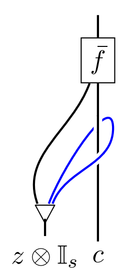
Lemma 5.15. *The identity morphism satisfies the triangle equality from Equation* (5.19).

Proof. The unitor for $\mathcal{Z}(\mathcal{A})_s$ is given in Lemma 2.18. Let $z \in \mathcal{Z}(\mathcal{A})$ and let $f\colon z \to \underset{\leftarrow}{\mathcal{C}}(c, c')$ be a morphism. The mate for the image of f under ρ is:



,

where we simplified a double symmetry between z and the summand of the strand, coming from the definition of the braiding on $\underset{\leftarrow}{\mathcal{C}}(c, c')$. On the other hand, the bottom route is the composite of f with 1_c, so in terms of mates becomes:



,

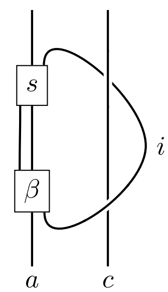
so the identity morphism indeed satisfies Equation (5.19). $\square$

$\mathcal{Z}(\mathcal{A})_s$-Tensoring

The category $\underset{\leftarrow}{\mathcal{C}}$ produced above is also tensored over $\mathcal{Z}(\mathcal{A})_s$.

Proposition 5.16. *Let $\mathcal{C}$ be a braided fusion category containing $\mathcal{A}$. Then for all $c, c' \in \mathcal{C}$ and $(a, \beta) \in \mathcal{Z}(\mathcal{A})$, the subobject $\Pi^\beta(ac)$ associated to the idempotent*

$$\Pi_{(a,\beta),c} = \sum_{i \in \mathcal{O}(\mathcal{A})}$$



satisfies

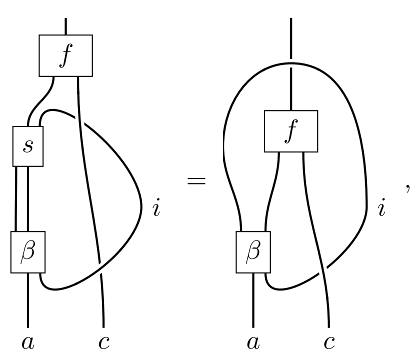
$$\mathcal{Z}(\mathcal{A})((a,\beta), \underset{\leftarrow}{\mathcal{C}}(c, c')) \cong \mathcal{C}(\Pi^\beta(ac), c').$$

Proof. We observe that $\Pi^\beta(ac)$ is well-defined, the idempotent $\Pi_{(a,\beta),c}$ commutes with the braiding by naturality of the braiding in $\mathcal{C}$.

To see the isomorphism, we notice that, in $\mathcal{Z}(\mathcal{A})$, the hom-spaces between (a, β) and (a', β') are the equalisers for

$$\begin{aligned} \mathcal{A}(a, a') \xrightarrow{\oplus \mathrm{coev}_i} \bigoplus_{i \in \mathcal{O}(\mathcal{A})} \mathcal{A}(i^* i a, a') \xrightarrow{\cong} \mathcal{A}(ia, ia') \xrightarrow{\oplus \beta_i^*, \beta_{i,*}^{-1}} \bigoplus_{i \in \mathcal{O}(\mathcal{A})} \mathcal{A}(ai, ia') \\ \xrightarrow{\cong} \bigoplus_{i \in \mathcal{O}(\mathcal{A})} \mathcal{A}(aii^*, a') \xrightarrow{\oplus \mathrm{ev}_i} \mathcal{A}(a, a') \end{aligned} \tag{5.20}$$

and the identity on $\mathcal{A}(a, a')$. The hom-object $\mathcal{C}(\Pi^\beta(ac), c')$ is the equaliser for precomposition with $\Pi_{(a,\beta),c}$ and the identity on $\mathcal{C}(ac, c')$. Precomposition with $\Pi_{(a,\beta),c}$ takes a morphism $f \colon ac \to c'$ to the morphism



where we have used naturality of the braiding in $\mathcal{C}$ on ac. Under the isomorphism

$$\mathcal{C}(ac, c') \cong \mathcal{A}(a, \underline{C}(c, c')),$$

this is sent to the composite from Equation (5.20). So we see that the left and right hand side are equalisers for the same morphisms, and therefore canonically isomorphic. $\square$

Monoidal structure

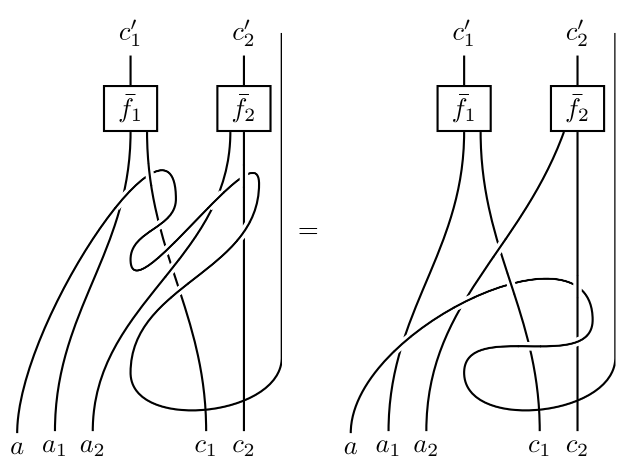
The monoidal structure on $\mathcal{C}$ will give rise to a monoidal structure on the $\mathcal{Z}(\mathcal{A})$-enriched version of $\mathcal{C}$. Unlike previously, this monoidal structure will not factor over the $(\mathcal{Z}(\mathcal{A}), \otimes_s)$-product, but will rather factor over a product where we combine both monoidal structures on $\mathcal{Z}(\mathcal{A})$. That is, it will be a $\mathcal{Z}(\mathcal{A})$-crossed tensor category (Definition 4.16).

Proposition 5.17. *If $\mathcal{C}$ is a braided tensor category containing a symmetric spherical fusion category $\mathcal{A}$, then $\underset{\leftarrow}{\mathcal{C}}$ is $\mathcal{Z}(\mathcal{A})$-crossed tensor (see Definition 4.16), with monoidal structure given in Definition 5.3 lifted to the $\mathcal{Z}(\mathcal{A})_s$-enriched category.*

Proof. As the monoidal structure from Definition 5.3 is compatible with the composition, and the composition in $\underset{\leftarrow}{\mathcal{C}}$ is a restriction of this, the lift of the monoidal structure will be compatible with composition. We still need to show that the morphisms

$$\underset{\leftarrow}{\mathcal{C}}(c_1, c_1') \otimes_c \underset{\leftarrow}{\mathcal{C}}(c_2, c_2') \xrightarrow{\otimes_{c_1 \boxtimes c_2, c_1' \boxtimes c_2'}} \underset{\leftarrow}{\mathcal{C}}(c_1 c_2, c_1' c_2')$$

are compatible with the braiding, so that they lift to $\mathcal{Z}(\mathcal{A})$. In $\underset{\leftarrow}{\mathcal{C}} \underset{c}{\boxtimes} \underset{\leftarrow}{\mathcal{C}}$, the left hand object will be equipped with the consecutive braiding on both factors, while the braiding on the right hand side comes from the braiding monodromy of $c_1 c_2$. Comparing these braidings with some $a \in \mathcal{A}$ in terms of mates for $f_1 \colon c_1 \to_{a_1} c_1'$ and $f_2 \colon c_2 \to_{a_2} c_2'$ gives



,

for first braiding and then applying $\otimes$ and the vice versa, respectively. These two sides are indeed equal. $\square$

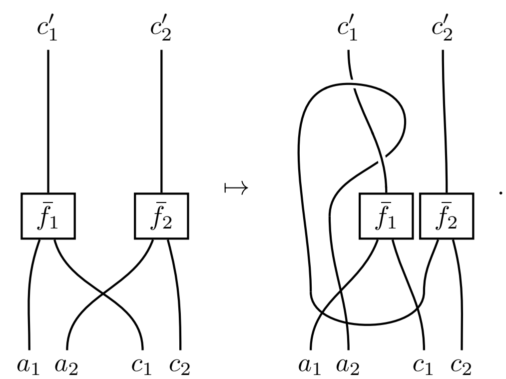
$\mathcal{Z}(\mathcal{A})$-crossed braiding

We will now show that the braiding for $\mathcal{C}$ gives rise to a $\mathcal{Z}(\mathcal{A})$-crossed braiding, see Definition 4.20. The first step is to examine what the braiding functor B from Definition 4.13 becomes for $\underleftarrow{\mathcal{C}} \boxtimes_{c} \underleftarrow{\mathcal{C}}$. To do this, we define:

Definition-Propostion 5.18. The functor $\beta^{-2}\colon \underleftarrow{\mathcal{C}} \boxtimes_{\mathcal{A}} \underleftarrow{\mathcal{C}} \to \underleftarrow{\mathcal{C}} \boxtimes_{\mathcal{A}} \underleftarrow{\mathcal{C}}$ is defined as follows. Note that we have the following isomorphisms:

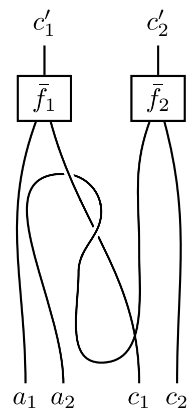
$$\begin{aligned} \underleftarrow{\mathcal{C}} \boxtimes_{\mathcal{A}} \underleftarrow{\mathcal{C}}(c_1 \boxtimes c_2, c_1' \boxtimes c_2') &= \underleftarrow{\mathcal{C}}(c_1, c_1') \otimes \underleftarrow{\mathcal{C}}(c_2, c_2') \\ &= \underleftarrow{\mathcal{C}}(c_1, \underleftarrow{\mathcal{C}}(c_2, c_2')c_1'). \end{aligned}$$

This last object has an automorphism induced by the inverse braiding monodromy of $\underleftarrow{\mathcal{C}}(c_2, c_2')$ and c_1'. In terms of mates for $f\colon c_1 \boxtimes c_2 \to_a c_1' \boxtimes c_2'$ factored over the tensor product $f_1\colon c_1 \to_{a_1} c_1'$ and $f_2\colon c_2 \to_{a_2} c_2'$ this becomes:



(5.21)

We remind the reader of the convention discussed around Equation (5.5), and emphasise that the double braiding in this diagram really is a double braiding, whereas the first crossing does not have any meaning. Unfortunately, this description of the action of the double braiding in terms of mates makes it unclear how to apply the isomorphism to get back to a morphism $c_1 \boxtimes c_2 \to_{a_1 a_2} c_1' \boxtimes c_2'$, as it is not manifestly of the form in Equation (5.5). To bring it into this form, we bring down the double braiding to get:



, (5.22)

which, comparing with (5.7), we can interpret as the mate for the tensor product of morphisms $a_1 a_2 a_2^* \to \underleftarrow{\mathcal{C}}(c_1, c_1')$ and $f_2 \colon c_2 \to_{a_2} c_2'$, with trivalent vertex $a_1 a_2 \to (a_1 a_2 a_2^*) \otimes a_2$.

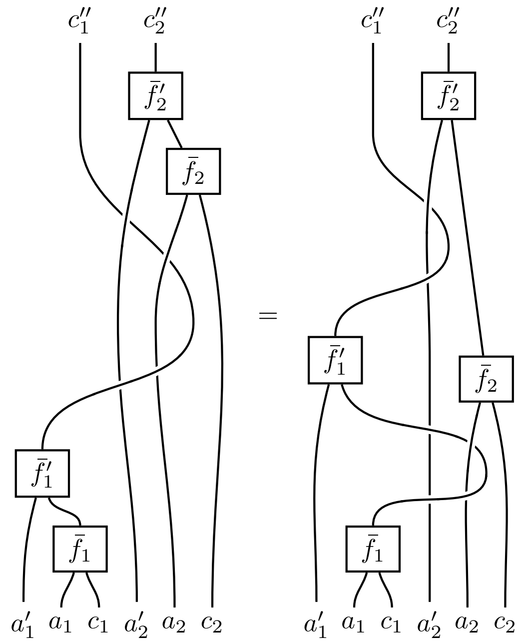
These automorphisms induced by the inverse braiding monodromies as above compile to an automorphism β^{-2} of $\underleftarrow{\mathcal{C}} \underset{\mathcal{A}}{\boxtimes} \underleftarrow{\mathcal{C}}$.

Proof. We check that the following diagram commutes

$$\begin{tikzcd}
\underleftarrow{\mathcal{C}}(c_1, c_1')\underleftarrow{\mathcal{C}}(c_2, c_2')\underleftarrow{\mathcal{C}}(c_1', c_1'')\underleftarrow{\mathcal{C}}(c_2', c_2'') \arrow[r, "\circ"] \arrow[d, "\beta^{-2}"] & \underleftarrow{\mathcal{C}}(c_1, c_1'')\underleftarrow{\mathcal{C}}(c_2, c_2'') \arrow[d, "\beta^{-2}"] \\
\underleftarrow{\mathcal{C}}(c_1, c_1')\underleftarrow{\mathcal{C}}(c_2, c_2')\underleftarrow{\mathcal{C}}(c_1', c_1'')\underleftarrow{\mathcal{C}}(c_2', c_2'') \arrow[r, "\circ"] & \underleftarrow{\mathcal{C}}(c_1, c_1'')\underleftarrow{\mathcal{C}}(c_2, c_2'').
\end{tikzcd}$$

In terms of mates for $f_1 \colon c_1 \to_{a_1} c_1'$, $f_1' \colon c_1' \to_{a_1'} c_1''$, $f_2 \colon c_2 \to_{a_2} c_2'$ and

$f_2' \colon c_2' \to_{a_2} c_2''$, this becomes:



(5.23)

The strand a_2' undercrosses the strand c_1', conform the conventions introduced. Using that a_1' and a_2 are transparent to each other, we see that these string diagrams are indeed equal. □

We can compute B in terms of this functor:

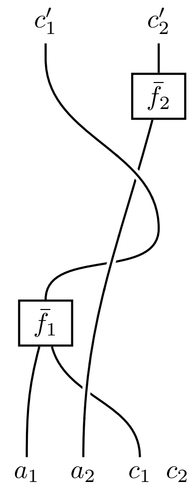
Lemma 5.19. *Let $\underleftarrow{\mathcal{C}}$ be as above. The braiding functor B (see Definition 4.13) on $\underleftarrow{\mathcal{C}} \boxtimes_c \underleftarrow{\mathcal{C}}$ is given by the composite of the functor β^{-2} (Definition 5.18) with the symmetry in $\mathcal{A}$.*

Proof. This is immediate from the definition of the half-braidings on the hom-objects (Definition 5.10). □

The braiding is by definition a natural transformation from $\otimes_{\underleftarrow{\mathcal{C}}}$ to $\otimes_{\underleftarrow{\mathcal{C}}} \circ B$. So, our next step is to compute the composite of B with the monoidal structure. It turns out that the resulting functor can be viewed as the monoidal structure $\otimes^{\beta}$ on $\underleftarrow{\mathcal{C}}$ from Definition 5.6. A similar argument to the proof of Proposition 5.17 shows that $\otimes^{\beta}$ defines a monoidal structure on $\underleftarrow{\mathcal{C}}$.

Lemma 5.20. *The functor $- \otimes^{\beta} -$ is equal to the functor obtained by precomposing $- \otimes_{\underleftarrow{\mathcal{C}}} -$ with β^{-2}.*

Proof. We only need to check the functors agree on morphisms, so let $f_1 \colon c_1 \to_{a_1} c_1'$ and $f_2 \colon c_2 \to_{a_2} c_2'$ be morphisms in $\underleftarrow{\mathcal{C}}$. Their image under $- \otimes^{\beta} -$ is shown in Equation (5.13). Their image under the composite of β^{-2} and $-\otimes-$ is given, in string diagrams, by:



, (5.24)

which is indeed equal to Equation (5.13), using that a_1 and a_2 are transparent with respect to each other. ☐

We will now show that the braiding is a natural transformation between these two monoidal structures on $\underleftarrow{\mathcal{C}}$, where we compose one with the switch map. We will then lift this result to $\underleftarrow{\underleftarrow{\mathcal{C}}}$.

Lemma 5.21. *The braiding in $\mathcal{C}$ induces a natural isomorphism between the functors $-\otimes^\beta -\colon \underleftarrow{\mathcal{C}} \boxtimes_{\mathcal{A}} \underleftarrow{\mathcal{C}} \to \underleftarrow{\mathcal{C}}$ and the composite of $-\otimes -\colon \underleftarrow{\mathcal{C}} \boxtimes_{\mathcal{A}} \underleftarrow{\mathcal{C}} \to \underleftarrow{\mathcal{C}}$ with the switch map for the $\mathcal{A}$-product. This isomorphism satisfies the hexagon equations.*

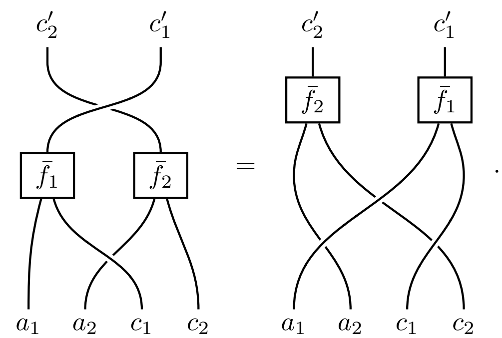
Proof. We want to show the diagram

$$\begin{tikzcd}
c_1c_2 \arrow[r, "\beta_{c_1,c_2}"] \arrow[d, "a_1a_2"', "f_1\otimes^\beta f_2"] & c_2c_1 \arrow[d, "a_2a_1"', "f_2\otimes f_1"] \\
c_1'c_2' \arrow[r, "\beta_{c_1',c_2'}"'] & c_2'c_1'
\end{tikzcd}$$

commutes for all $f_1\colon c_1 \to_{a_1} c_1'$ and $f_2\colon c_2 \to_{a_2} c_2'$. In terms of the mates, this diagram becomes:

$$\begin{tikzcd}
a_1a_2c_1c_2 \arrow[r, "\beta_{a_1,a_2}\otimes\beta_{c_1,c_2}"] \arrow[d, "\beta^{-1}_{a_2,c_1}"] & a_2a_1c_2c_1 \arrow[d, "\beta_{c_2,a_1}"] \\
a_1c_1a_2c_2 \arrow[d, "\bar{f}_1\otimes\bar{f}_2"] & a_2c_2a_1c_1 \arrow[d, "\bar{f}_2\otimes\bar{f}_1"] \\
c_1'c_2' \arrow[r, "\beta_{c_1',c_2'}"'] & c_2'c_1'
\end{tikzcd} .$$

Writing this in terms of string diagrams:



The hexagon equations follow from the hexagon equations for the braiding in $\mathcal{C}$. ☐

Proposition 5.22. *Let $\mathcal{C}$ be a braided tensor category containing a spherical symmetric fusion category $\mathcal{A}$. Then the category $\underleftarrow{\mathcal{C}}$ is a $\mathcal{Z}(\mathcal{A})$-crossed braided tensor category.*

Proof. We have already shown in Proposition 5.17 that $\underleftarrow{\mathcal{C}}$ is $\mathcal{Z}(\mathcal{A})$-crossed tensor. We have to show that the braiding for $\mathcal{C}$ gives a natural transformation between the tensor structure and the composite of B (Definition 4.13) with the tensor structure. We know that B computes as $\beta^{-2} \circ \mathrm{Switch}_{\mathcal{A}}$, by Lemma 5.19. So we see that the composite

$$\otimes \circ B = \otimes \circ \beta^{-2} \circ \mathrm{Switch}_{\mathcal{A}} = \otimes^{\beta} \circ \mathrm{Switch}_{\mathcal{A}},$$

where the last equality is Lemma 5.20. But, by Proposition 5.21, the braiding in $\mathcal{C}$ induces a natural transformation between this functor and $\otimes$. ☐

Enriching the Commutant of $\mathcal{A}$

We will now for a category $\underleftarrow{\mathcal{C}}$ obtained by the enriching procedure above, give a characterisation of the neutral subcategory (see Definition 4.23) $\underleftarrow{\mathcal{C}}_{\mathcal{A}}$ in terms of the so-called braided commutant of $\mathcal{A}$ in $\mathcal{C}$.

Definition 5.23. Let $\mathcal{C}$ be a braided fusion category with braiding β and let $\mathcal{B}$ be a braided monoidal full subcategory. Then the *braided commutant of $\mathcal{B}$ in $\mathcal{C}$* is the full subcategory with objects

$$\mathcal{Z}_2(\mathcal{B},\mathcal{C}) = \{c \in \mathcal{C} | \beta_{c,b} \circ \beta_{b,c} = \mathrm{id}_{bc} \quad \forall b \in \mathcal{B}\}.$$

When $\mathcal{B} = \mathcal{C}$, we will denote this by $\mathcal{Z}_2(\mathcal{C})$. This $\mathcal{Z}_2(\mathcal{C})$ is called the *Müger centre of $\mathcal{C}$*.

When $\mathcal{A}$ is a symmetric subcategory of $\mathcal{C}$ the commutant $\mathcal{Z}_2(\mathcal{A},\mathcal{C})$ contains $\mathcal{A}$.

Proposition 5.24. *Denote by $\underleftarrow{\mathcal{Z}_2(\mathcal{A},\mathcal{C})} \subset \underleftarrow{\mathcal{C}}$ the full subcategory on the objects of $\mathcal{Z}_2(\mathcal{A},\mathcal{C})$. Then:*

$$\underleftarrow{\mathcal{Z}_2(\mathcal{A},\mathcal{C})} = \mathcal{C}_{\mathcal{A}}.$$

Proof. As this is a statement about small full subcategories, it suffices to show that $\underleftarrow{\mathcal{Z}_2(\mathcal{A},\mathcal{C})} \subset \mathcal{C}_{\mathcal{A}}$ and $\underleftarrow{\mathcal{Z}_2(\mathcal{A},\mathcal{C})} \supset \mathcal{C}_{\mathcal{A}}$ at the level of objects.

The inclusion $\underleftarrow{\mathcal{Z}_2(\mathcal{A},\mathcal{C})} \subset \mathcal{C}_{\mathcal{A}}$ follows directly from the way the half-braidings on $\underleftarrow{\mathcal{C}}(c,c')$ are defined in Definition 5.10: in Equation (5.15) the morphism $(\beta^{-1}_{a,c'}\beta^{-1}_{c',a})_*$ is just the identity, so the composite becomes the symmetry in $\mathcal{A}$ between $\underleftarrow{\mathcal{C}}(c,c')$ and a.

For the reverse inclusion, suppose that c is such that its Yoneda embedding $\underleftarrow{\mathcal{C}}(-,c)$ factors through $\mathcal{A}$. This means that for each $c' \in \mathcal{C}$, the hom-object $\underleftarrow{\mathcal{C}}(c',c)$ is $\underleftarrow{\mathcal{C}}(c',c)$ equipped with the symmetry in $\mathcal{A}$. Looking at the definition (Equation (5.15)) of the half-braiding, we see that this implies that $(\beta^{-1}_{a,c'}\beta^{-1}_{c',a})_*$ is the identity on $\underleftarrow{\mathcal{C}}(c',c)$ for all c'. By the Yoneda lemma this means that $\beta^{-1}_{a,c'}\beta^{-1}_{c',a}$ is the identity on ac, which is what we wanted to show. □

We observe the following, which is immediate from the above proposition combined with the fact that the composite $\mathcal{A} \hookrightarrow \mathcal{Z}(\mathcal{A}) \xrightarrow{\Phi} \mathcal{A}$ of the forgetful functor with the inclusion functor is the identity on $\mathcal{A}$:

Corollary 5.25. *Let $\mathcal{C}$ be a braided fusion category containing $\mathcal{A}$, and assume that $\mathcal{Z}_2(\mathcal{A},\mathcal{C}) = \mathcal{C}$. Then:*

$$\overline{\underleftarrow{\mathcal{C}}} = \underleftarrow{\mathcal{C}},$$

where $\overline{\mathcal{K}}$ for a $\mathcal{Z}(\mathcal{A})_s$-enriched category $\mathcal{K}$ was introduced in Definition 4.2.

5.2 De-enriching

5.2.1 The De-Enriching 2-Functor

Lemma 5.26. *The functor*

$$\mathcal{A}(\mathbb{I}_{\mathcal{A}}, -)\colon \mathcal{A} \to \mathbf{Vect}$$

is symmetric lax monoidal, with lax structure given by:

$$\mathcal{A}(\mathbb{I}_{\mathcal{A}}, \mathbb{I}_{\mathcal{A}}) \cong \mathbb{I}_{\mathbf{Vect}}$$
$$\mu_{a,a'}\colon \mathcal{A}(\mathbb{I},a)\mathcal{A}(\mathbb{I},a') \xrightarrow{\otimes_{\mathcal{A}}} \mathcal{A}(\mathbb{I}\mathbb{I},aa') \xrightarrow[\cong]{(\mathbb{I}\to\mathbb{I}\mathbb{I})^*} \mathcal{A}(\mathbb{I},aa'),$$

for $a, a' \in \mathcal{A}$.

Proof. As the unit part of the lax structure is an isomorphism, there is nothing to check there. To check compatibility of $\mu_{a',a''}$ with the associators, let $a, a', a'' \in \mathcal{A}$. We need to show that:

$$\begin{tikzcd}
\mathcal{A}(\mathbb{I},a)\mathcal{A}(\mathbb{I},a')\mathcal{A}(\mathbb{I},a'') \arrow[r, "\mu_{a',a''}"] \arrow[d, "\mu_{a,a'}"] & \mathcal{A}(\mathbb{I},a)\mathcal{A}(\mathbb{I},a'a'') \arrow[d, "\mu_{a,a'a''}"] \\
\mathcal{A}(\mathbb{I},aa')\mathcal{A}(\mathbb{I},a'') \arrow[r, "\mu_{aa',a''}"] & \mathcal{A}(\mathbb{I},aa'a''),
\end{tikzcd}$$

commutes. Using the definition of μ, this is equivalent to checking that

$$\begin{tikzcd}
\mathcal{A}(\mathbb{I},a)\mathcal{A}(\mathbb{I},a')\mathcal{A}(\mathbb{I},a'') \arrow[r, "\otimes_{\mathcal{A}}"] \arrow[d, "\otimes_{\mathcal{A}}"] & \mathcal{A}(\mathbb{I},a)\mathcal{A}(\mathbb{II},a'a'') \arrow[d, "\otimes_{\mathcal{A}}"] \\
\mathcal{A}(\mathbb{II},aa')\mathcal{A}(\mathbb{I},a'') \arrow[r, "\otimes_{\mathcal{A}}"] & \mathcal{A}(\mathbb{III},aa'a'')
\end{tikzcd}$$

commutes, but this is a direct consequence of the associativity of $\otimes_{\mathcal{A}}$. To see that this is a symmetric functor, consider:

$$\begin{tikzcd}
\mathcal{A}(\mathbb{I},a)\mathcal{A}(\mathbb{I},a') \arrow[r, "s_{\mathbf{Vect}}"] \arrow[d, "\otimes_{\mathcal{A}}"] & \mathcal{A}(\mathbb{I},a')\mathcal{A}(\mathbb{I},a) \arrow[d, "\otimes_{\mathcal{A}}"] \\
\mathcal{A}(\mathbb{II},aa') \arrow[r, "(s_{\mathcal{A}})_*"] \arrow[d, "\cong"] & \mathcal{A}(\mathbb{II},a'a) \arrow[d, "\cong"] \\
\mathcal{A}(\mathbb{I},aa') \arrow[r, "(s_{\mathcal{A}})_*"] & \mathcal{A}(\mathbb{I},a'a),
\end{tikzcd}$$

the bottom square commutes by naturality of the symmetry and its compatibility with the unitors, and the top square commutes as this is what it means for the symmetry to be a natural transformation between $\otimes$ and $\otimes \circ \mathrm{Switch}$. $\square$

Definition 5.27. Let $\mathcal{K} \in \mathcal{A}$**LinCat**. The *de-enrichment* of $\mathcal{K}$ is the linear category obtained from $\mathcal{K}$ by changing basis along the symmetric lax monoidal functor $\mathcal{A}(\mathbb{I}_{\mathcal{A}}, -)\colon \mathcal{A} \to \mathbf{Vect}$. This is the category with objects those of $\mathcal{K}$, and where the morphisms between two such objects a and b are $\mathcal{K}(a,b)$, c.f. Notation A.1.

If the category $\mathcal{K}$ is $\mathcal{Z}(\mathcal{A})_s$-enriched, then de-enriching is the composite of this construction with change of basis along the forgetful functor $\mathcal{Z}(\mathcal{A}) \to \mathcal{A}$.

Notation 5.28. We will denote by **LinCat** the 2-category $\mathcal{A}$**LinCat** with $\mathcal{A} = \mathbf{Vect}$. Its objects are just linear categories, and its **Vect**-product, that will be denoted by $\boxtimes$, is the Cauchy completion of the familiar Deligne tensor product of linear categories.

The following is a consequence of the general statements about change of basis, Propositions A.24 and A.31.

Corollary 5.29. *For all symmetric fusion categories $\mathcal{A}$, de-enrichment induces a bifunctor*

$$\mathbf{DeEnrich}\colon \mathcal{A}\mathbf{LinCat} \to \mathbf{LinCat}.$$

The composite of de-enrichment with the forgetful functor $\mathcal{Z}(\mathcal{A}) \to \mathcal{A}$ induces a bifunctor

$$\mathbf{DeEnrich}(\overline{(-)})\colon \mathcal{Z}(\mathcal{A})\mathbf{LinCat} \to \mathbf{LinCat}.$$

Using Proposition A.27 we see:

Corollary 5.30. *Let $\mathcal{K}$ be $\mathcal{A}$-monoidal, then $\mathbf{DeEnrich}(\mathcal{K})$ is a tensor category.*

Similarly, we have:

Lemma 5.31. *Let $\mathcal{K}$ be braided $\mathcal{A}$-monoidal, then $\mathbf{DeEnrich}(\mathcal{K})$ is a braided tensor category.*

Definition 5.32. We let the *comparison functor*

$$M\colon \mathbf{DeEnrich}(\overline{\mathcal{K}}) \boxtimes \mathbf{DeEnrich}(\overline{\mathcal{L}}) \to \mathbf{DeEnrich}(\overline{\mathcal{K} \underset{c}{\boxtimes} \mathcal{L}})$$

be the functor obtained from Lemma A.26 together with Lemma 4.19.

Corollary 5.33. *Let $\mathcal{K}$ be $\mathcal{Z}(\mathcal{A})$-crossed tensor, then $\mathbf{DeEnrich}(\overline{\mathcal{K}})$ is a tensor category, with tensor stucture given by:*

$$\mathbf{DeEnrich}(\overline{\mathcal{K}})\boxtimes\mathbf{DeEnrich}(\overline{\mathcal{K}}) \xrightarrow{M} \mathbf{DeEnrich}(\overline{\mathcal{K} \underset{c}{\boxtimes} \mathcal{K}}) \xrightarrow{\mathbf{DeEnrich}(\overline{\otimes})} \mathbf{DeEnrich}(\overline{\mathcal{K}}).$$

Proof. This combines Lemma 4.19 with Corollary 5.30. $\square$

Lemma 5.34. *Let B be as in Definition 4.13. The linear functor $\mathbf{DeEnrich}(\overline{B})$ fits into the following commutative diagram with the switch map* Switch *in* **LinCat***:*

$$\begin{tikzcd}
\mathbf{DeEnrich}(\overline{\mathcal{K}}) \boxtimes \mathbf{DeEnrich}(\overline{\mathcal{L}}) \arrow[r, "M"] \arrow[d, "\mathrm{Switch}"] & \mathbf{DeEnrich}(\overline{\mathcal{K} \underset{c}{\boxtimes} \mathcal{L}}) \arrow[d, "\mathbf{DeEnrich}(\overline{B})"] \\
\mathbf{DeEnrich}(\overline{\mathcal{L}}) \boxtimes \mathbf{DeEnrich}(\overline{\mathcal{K}}) \arrow[r, "M"] & \mathbf{DeEnrich}(\overline{\mathcal{L} \underset{c}{\boxtimes} \mathcal{K}})
\end{tikzcd}$$

Proof. Both routes are the same on objects, so we have to check that:

$$\begin{tikzcd}
\mathcal{K}(k,k')\mathcal{L}(l,l') \arrow[r, "\mu"] \arrow[d, "\mathrm{switch}"] & \mathbf{DeEnrich}(\underline{\mathcal{K}}(k,k') \otimes_c \underline{\mathcal{L}}(l,l')) \arrow[d, "\beta"] \\
\mathcal{L}(l,l')\mathcal{K}(k,k') \arrow[r, "\mu"] & \mathbf{DeEnrich}(\underline{\mathcal{L}}(l,l') \otimes_c \underline{\mathcal{K}}(k,k')),
\end{tikzcd}$$

commutes. De-enriching just picks out the unit summands of the right hand column, and μ is inclusion. Recall that any half braiding $\beta\colon -\otimes a \Rightarrow a \otimes -$ for an object $a \in \mathcal{A}$ satisfies $\beta_{\mathbb{I}} = \rho^{-1} \circ \lambda = s_{\mathbb{I},a}$. This means that β agrees with the switch in vector spaces on the image of μ, which is what we wanted to show. $\square$

Proposition 5.35. *Let $\mathcal{K}$ be a $\mathcal{Z}(\mathcal{A})$-crossed braided $\mathcal{A}$-monoidal category, then* $\mathbf{DeEnrich}(\overline{\mathcal{K}})$ *is a braided tensor category.*

Proof. From Corollary 5.33 we read off that the braiding would have to be a natural isomorphism filling in the following diagram:

$$\begin{tikzcd}
\mathbf{DeEnrich}(\overline{\mathcal{K}}) \boxtimes \mathbf{DeEnrich}(\overline{\mathcal{K}}) \arrow[rr, "\mathrm{Switch}"] \arrow[d, "M"'] & & \mathbf{DeEnrich}(\overline{\mathcal{K}}) \boxtimes \mathbf{DeEnrich}(\overline{\mathcal{K}}) \arrow[d, "M"] \\
\mathbf{DeEnrich}(\overline{\mathcal{K} \underset{c}{\boxtimes} \mathcal{K}}) \arrow[rr, "\mathbf{DeEnrich}(\overline{B})"] \arrow[dr, "\mathbf{DeEnrich}(\otimes)"'] & & \mathbf{DeEnrich}(\overline{\mathcal{K} \underset{c}{\boxtimes} \mathcal{K}}) \arrow[dl, "\mathbf{DeEnrich}(\otimes)"] \\
& \mathbf{DeEnrich}(\overline{\mathcal{K}}). &
\end{tikzcd}$$

We recognise the top square as the square from Lemma 5.34, and the bottom square is the image under $\mathbf{DeEnrich}\overline{(-)}$ of the defining triangle for the braiding (Definition 4.20). This means the image under $\mathbf{DeEnrich}\overline{(-)}$ of the braiding for $\mathcal{K}$ will give a braiding for $\mathbf{DeEnrich}(\overline{\mathcal{K}})$. □

5.2.2 Equivalence between braided categories containing $\mathcal{A}$ and $\mathcal{Z}(\mathcal{A})$-crossed braided categories

The goal of this section is to show that the construction $\mathcal{C} \mapsto \underset{\leftarrow}{\mathcal{C}}$ outlined in Section 5.1 above gives an equivalence of 2-categories between a 2-category $\mathbf{BFC}/\mathcal{A}$ of braided fusion categories containing $\mathcal{A}$ and $\mathcal{Z}(\mathcal{A})-\mathbf{XBT}$ (see Definition 4.26), with inverse given by $\mathbf{DeEnrich}\overline{(-)}$. In defining the 2-category $\mathbf{BFC}/\mathcal{A}$, there are several choices to be made, we use the following definition:

Definition 5.36. The *2-category of braided fusion categories containing the symmetric fusion category* $\mathcal{A}$ $\mathbf{BFC}/\mathcal{A}$ is the 2-category with

- objects: braided fusion categories $\mathcal{C}$ with a braided monoidal embedding $\mathcal{A} \subset \mathcal{C}$,
- morphisms: braided monoidal functors that restrict to the identity on $\mathcal{A}$,
- 2-morphisms: monoidal natural transformations that restrict to the identity natural transformation on $\mathcal{A}$.

Remark 5.37. There are several ways of making this definition less restrictive. First of all, we can allow natural transformations to be something else than just the identity on $\mathcal{A}$. Second of all, we could ask for the restriction of the functors to $\mathcal{A}$ to be a self-equivalence of $\mathcal{A}$, or even an arbitrary endofunctor. The definition given here is the one that fits with Definition 4.26.

Our first goal is to extend $\mathcal{C} \mapsto \underset{\leftarrow}{\mathcal{C}}$ to a 2-functor, as so far we have only defined it on the objects of $\mathbf{BFC}/\mathcal{A}$. To specify what it does on functors:

Definition-Propostion 5.38. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be morphism in $\mathbf{BFC}/\mathcal{A}$, then *the associated $\mathcal{Z}(\mathcal{A})$-enriched functor*

$$\underleftarrow{F}\colon \underleftarrow{\mathcal{C}} \to \underleftarrow{\mathcal{C}}',$$

is the functor which acts as F on objects. On morphisms, we first define the morphisms $\underleftarrow{F}_{c,c'} \in \mathcal{A}$ by observing that the composite:

$$\mathcal{C}(ac, c') \xrightarrow{F_{ac,c'}} \mathcal{C}'(F(ac), F(c')) \cong \mathcal{C}'(aF(c), F(c')),$$

where the last isomorphism comes from the monoidality of F and the fact that F is the identity on $\mathcal{A}$, gives for each $c, c' \in \mathcal{C}$ a natural transformation from $\mathcal{C}(-c, c')\colon \mathcal{A} \to \mathbf{Vect}$ to $\mathcal{C}'(-F(c), F(c'))\colon \mathcal{A} \to \mathbf{Vect}$. This natural transformation induces a morphism:

$$\underleftarrow{F}_{c,c'}\colon \underleftarrow{\mathcal{C}}(c, c') \to \underleftarrow{\mathcal{C}}'(Fc, Fc').$$

This morphism takes the mate $\bar{f}\colon ac \to a$ for a morphism $f\colon c \to_a c'$ to $\underleftarrow{F}(\bar{f})\colon aF(c) \xrightarrow{\cong} F(ac) \xrightarrow{\bar{f}} F(c')$, where the first map is the monoidality isomorphism for F.

This $\underleftarrow{F}_{c,c'}$ commutes with the braidings on $\underleftarrow{\mathcal{C}}(c, c')$ and $\underleftarrow{\mathcal{C}}'(Fc, Fc')$ for each $c, c' \in \mathcal{C}$, and combine to a $\mathcal{Z}(\mathcal{A})_s$-enriched functor. Furthermore, $\underleftarrow{F}$ is braided monoidal and is compatible with the $\mathcal{A}$-tensoring (c.f. Definition 4.26).

Proof. We start by observing that, as F is braided, the morphism $\underleftarrow{F}_{c,c'}$ will be compatible with the braiding and therefore define a morphism $\underleftarrow{F}_{c,c'}$ in $\mathcal{Z}(\mathcal{A})$. These morphisms will combine to an $\mathcal{Z}(\mathcal{A})_s$-enriched functor, to see that it preserves composition, it is enough to show that for all $c, c', c'' \in \mathcal{C}$:

$$\begin{tikzcd}
\underleftarrow{\mathcal{C}}(c, c')\underleftarrow{\mathcal{C}}(c', c'') \arrow[r, "\circ"] \arrow[d, "\underleftarrow{F}_{c,c'} \otimes \underleftarrow{F}_{c',c''}"] & \underleftarrow{\mathcal{C}}(c, c'') \arrow[d, "\underleftarrow{F}_{c,c''}"] \\
\underleftarrow{\mathcal{C}}'(Fc, Fc')\underleftarrow{\mathcal{C}}'(Fc', Fc'') \arrow[r, "\circ"] & \underleftarrow{\mathcal{C}}'(Fc, Fc'').
\end{tikzcd}$$

On mates for $f\colon c \to_a c'$ and $f'\colon c' \to_{a'} c''$, the top route computes as:

$$a'aFc \xrightarrow{\cong} F(a'ac) \xrightarrow{\bar{f}' \circ (\mathrm{id}_{a'} \otimes \bar{f})} F(c''),$$

whereas the bottom route becomes:

$$a'aFc \xrightarrow{\cong} a'F(ac) \xrightarrow{\mathrm{id}_{a'} F(\bar{f})} a'F(c') \xrightarrow{\cong} F(a'c') \xrightarrow{F(\bar{f}')} F(c'').$$

Using the fact that the monoidality isomorphism for F is natural, we can exchange the middle two morphisms to get:

$$a'aF(c) \xrightarrow{\cong} F(a'ac) \xrightarrow{F(\bar{f}' \circ (\mathrm{id}_a \otimes \underleftarrow{F}(\bar{f})))} F(c''),$$

where we have also used the fact that F preserves composition, and that the monoidality isomorphisms for $aF(c)$ and $a'F(ac)$ compose to the monoidality isomorphism for $F(a'ac)$.

Finally, we observe that as F preserves the braiding on $\mathcal{C}$, it will also preserve the braiding on $\underleftarrow{\mathcal{C}}$. $\square$

On the natural transformations, we use the following:

Definition-Propostion 5.39. Let $\kappa \colon F \Rightarrow G$ be a 2-morphism between two morphisms in $\mathbf{BFC}/\mathcal{A}$ between $\mathcal{C}$ and $\mathcal{C}'$. Then the *associated enriched natural transformation* $\underleftarrow{\kappa} \colon \underleftarrow{F} \Rightarrow \underleftarrow{G}$ is given by the mate to κ. This natural transformation satisfies the condition from Definition 4.26.

Proof. The condition from Definition 4.26 follows directly from the fact that κ is monoidal and restricts to the identity on $\mathcal{A}$. $\square$

With these assignments, we can now define:

Definition-Propostion 5.40. The assignment $\underleftarrow{\underline{(-)}}$ defines a bifunctor

$$\underleftarrow{\underline{(-)}} \colon \mathbf{BFC}/\mathcal{A} \to \mathcal{Z}(\mathcal{A}) - \mathbf{XBT}.$$

Proof. We have to check that this assignment preserves composition of functors and of natural transformations. The composite of mates is the mate of composites for degree $\mathbf{I}_s$-morphisms, and similarly, as the action of a composition of functors on hom-objects is by the composition of the maps the functors induce on hom-objects, the image of the composition under $\underleftarrow{\underline{(-)}}$ will be the composite of the images. $\square$

Theorem 5.41. *The bifunctors* $\mathbf{DeEnrich}\overline{(-)}$ *and* $\underleftarrow{\underline{(-)}}$ *are mutually inverse.*

Proof. The composite $\mathbf{DeEnrich}\overline{(-)} \circ \underleftarrow{\underline{(-)}}$ is clearly the identity. For the other composite, we observe that, for $\mathcal{K} \in \mathcal{Z}(\mathcal{A})$-$\mathbf{XBF}$, the underlying objects in $\mathcal{A}$ of the hom-objects of $\underleftarrow{\underline{\mathbf{DeEnrich}(\overline{\mathcal{K}})}}$ are characterised by:

$$\begin{aligned}
\mathcal{A}(a, \overline{\underleftarrow{\underline{\mathbf{DeEnrich}(\overline{\mathcal{K}})}}}(k,k')) &\cong \mathbf{DeEnrich}(\overline{\mathcal{K}})(ak,k') \\
&= \mathcal{A}(\mathbb{I}_{\mathcal{A}}, \overline{\mathcal{K}}(ak,k')) \\
&\cong \mathcal{A}(\mathbb{I}_{\mathcal{A}}, a^*\overline{\mathcal{K}}(k,k')) \\
&\cong \mathcal{A}(a, \overline{\mathcal{K}}(k,k')),
\end{aligned}$$

where the first isomorphism is the $\mathcal{A}$-tensoring (Definition 4.2) of $\underleftarrow{\underline{\mathbf{DeEnrich}(\overline{\mathcal{K}})}}$, the second the definition of $\mathbf{DeEnrich}$, the third is Lemma A.13, and the final equality is the adjunction between $\otimes a$ and $a^*\otimes$ on $\mathcal{A}$. These isomorphisms are all natural, so will combine to an equivalence between $\underleftarrow{\mathbf{DeEnrich}(\overline{\mathcal{K}})}$ and $\mathcal{K}$, as long as we can show that they lift to morphisms in $\mathcal{Z}(\mathcal{A})$. To see this, observe that the half-braidings on $\underleftarrow{\mathbf{DeEnrich}(\overline{\mathcal{K}})}(k, k')$ are both defined in terms of the braiding of k with objects in $\mathcal{A}$. $\square$

Chapter 6

(De)-Equivariantisation and the Reduced Tensor Product

In this chapter, we will define the reduced tensor product of braided fusion categories over a symmetric fusion category. To examine its properties, such as what the invertible objects for this product are, it will be useful to examine how the equivalences between **BFC**/$\mathcal{A}$, $\mathcal{Z}(\mathcal{A})$-**XBF** and G-**XBF** (or (G,ω)-**XBF**) interact. We will show that the composite

$$\mathcal{C} \mapsto \mathbf{DeEnrich}(\overline{\mathbf{Fix}(\mathcal{C})}),$$

corresponds to equivariantisation [DGNO10], whereas

$$\mathcal{C} \mapsto \underleftarrow{\underline{\overline{\overline{\mathcal{C}}}}}$$

corresponds to de-equivariantisation [DGNO10]. After having set up the reduced tensor product, we take a look at its properties and collect directions for future work.

6.1 (De-)Equivariantisation

In this section we discuss the relation between the constructions done in the previous Chapters 4.2 and 5, and (de-)equivariantisiation. After this, we define the reduced tensor product.

6.1.1 (De-)Equivariantisation and $\mathcal{Z}(\mathcal{A})$-Crossed Categories

In this section, we will focus on the case where the symmetric subcategory $\mathcal{A} \cong \mathrm{Rep}(G)$ is Tannakian.

(De-)Equivariantisation as 2-Functors

De-equivariantisation and equivariantisation define a pair of mutually inverse 2-functors:

Theorem 6.1 ([DGNO10]). *There is an equivalence of 2-categories:*

$$\mathbf{De}-\mathbf{Eq}\colon \mathbf{BFC}/\mathcal{A} \longleftrightarrow G\text{-}\mathbf{XBF}\colon \mathbf{Eq},$$

here $\mathbf{BFC}/\mathcal{A}$ *is as in Definition 5.36,* G-$\mathbf{XBF}$ *is defined in Definition 4.34, and* $\mathbf{Eq}$ *is the equivariantisation functor from Definition 6.3. De-equivariantisation* $\mathbf{De}-\mathbf{Eq}$ *is defined in [DGNO10].*

Remark 6.2. While we will recall the definition of equivariantisation in Definition 6.3 below, we will not need the definition of de-equivariantisation, we just need to know it is the inverse to **Eq**.

Relation to $\mathcal{Z}(\mathcal{A})$-Crossed Braided Categories

We will prove that the following diagram of 2-functors commutes:

$$\begin{tikzcd}
 & \mathcal{Z}(\mathcal{A})\text{-}\mathbf{XBF} \arrow[dl, shift left, "\mathbf{DeEnrich}\overline{(-)}"] \arrow[dr, shift left, "\overline{\overline{(-)}}"] & \\
\mathbf{BFC}/\mathcal{A} \arrow[ur, shift left, "\underleftarrow{(-)}"] \arrow[rr, shift left, "\mathbf{De-Eq}"] & & G\text{-}\mathbf{XBF}. \arrow[ul, shift left, "\mathbf{Fix}"] \arrow[ll, shift left, "\mathbf{Eq}"]
\end{tikzcd}$$

This splits into two parts. First we will show that the diagram commutes for the left-pointing arrows. From this, we will deduce that the diagram for the right-pointing arrows also commutes.

Factorising Equivariantisation

We will now show that the composite of **Fix** with $\mathbf{DeEnrich}\overline{(-)}$ is the equivariantisation 2-functor **Eq**.

We first recall the definition of **Eq**.

Definition 6.3 ([Müg10]). Let $\mathcal{C}$ be a G-crossed braided fusion category. Then the equivariantisation (or homotopy fixed point category) is the braided fusion category $\mathbf{Eq}(\mathcal{C})$ with objects pairs (c,u), where $u=\{u_g\}_{g\in G}$ is a set of isomorphisms $u_g\colon c^g\to c$, satisfying $u_g(u_h)^g=u_{gh}$.

The hom-spaces are given by:

$$\mathbf{Eq}(\mathcal{C})((c,u),(c',u'))=\mathcal{C}(c,c')^G,$$

where the superscript G denotes taking the invariants for the G-action on $\mathcal{C}(c,c')$ given by:

$$\mathcal{C}(c,c')\xrightarrow{(-)^g}\mathcal{C}(c^g,(c')^g)\xrightarrow{(u_g)^*,(u'_g)^{-1}_*}\mathcal{C}(c,c').$$

The tensor product and braiding are inherited from $\mathcal{C}$. For F a functor of G-crossed braided categories, $\mathbf{Eq}(F)$ is defined by taking pairs (c, u) to their image under F. As functors of G-crossed braided categories intertwine the G-action, this extends $\mathbf{Eq}(F)$ to a functor. We similarly define $\mathbf{Eq}$ on natural transformations.

This definition is similar to Definition 4.52. In fact, one has:

Theorem 6.4. *The composite of 2-functors*

$$G\text{-}\mathbf{XBF} \xrightarrow{\mathbf{Fix}} \mathcal{Z}(\mathcal{A})\text{-}\mathbf{XBF} \xrightarrow{\mathbf{DeEnrich}\overline{(-)}} \mathbf{BFC}/\mathcal{A}$$

is equal to $\mathbf{Eq}$.

Proof. We will only provide a proof of this statement on objects of the 2-categories. The proof for 1-morphisms and 2-morphisms proceeds analogously.

Let $\mathcal{C}$ be a G-crossed braided fusion category. We want to show that $\mathbf{DeEnrich}(\overline{\mathbf{Fix}(\mathcal{C})})$ is equal to $\mathbf{Eq}(\mathcal{C})$. We observe that, by definition, their objects are equal.

Furthermore, the G-action on the G-equivariant vector bundle $\mathbf{Fix}(\mathcal{C})((c, u), (c', u'))$, which has underlying vector space $\mathcal{C}(c, c')$, is the G-action with respect to which we take G-invariants in Definition 6.3.

We therefore need to show that $\mathbf{DeEnrich}\overline{(-)}$ takes G-invariants on hom-objects. Recall that $\overline{(-)}$ is the operation of applying the forgetful functor $\mathcal{Z}(\mathcal{A}) \to \mathcal{A}$ to hom-objects. This will take $\mathbf{Fix}(\mathcal{C})((c, u), (c', u'))$ to the G-representation with underlying vector space $\mathcal{C}(c, c')$ and G-action from Definition 6.3. De-enriching is then applying $\mathcal{A}(\mathbb{I}_{\mathcal{A}}, -)$ to hom-objects, this indeed picks out the G-invariants, as $\mathbb{I}_{\mathcal{A}}$ is the trivial representation. □

Factorising De-Equivariantisation

Since we have shown that the left-pointing arrows in the diagram in 6.1.1 form a commutative diagram, and all the pairs of arrows are mutual inverses, we have:

Corollary 6.5. *The composite of 2-functors*

$$\mathbf{BFC}/\mathcal{A} \xrightarrow{\underleftarrow{\underline{(-)}}} \mathcal{Z}(\mathcal{A})\text{-}\mathbf{XBF} \xrightarrow{\overline{\overline{\overline{(-)}}}} G\text{-}\mathbf{XBF}$$

is equivalent to de-equivariantisation.

6.2 Reduced Tensor Product

In this section we discuss the reduced tensor product. We first discuss its definition and some of its basic properties, and then move on to some applications.

6.2.1 Definition and Properties of the Reduced Tensor Product

Definition of the Reduced Tensor Product

Definition 6.6. The *reduced tensor product* $\overset{\mathcal{A}}{\underset{\text{red}}{\boxtimes}}$ on the category $\mathbf{BFC}/\mathcal{A}$ (see Definition 5.36) is defined to be the composite:

$$\begin{tikzcd}
\mathbf{BFC}/\mathcal{A} \times \mathbf{BFC}/\mathcal{A} \arrow[r, "\underleftarrow{(-)} \times \underleftarrow{(-)}"] & \mathcal{Z}(\mathcal{A})\text{-}\mathbf{XBF} \times \mathcal{Z}(\mathcal{A})\text{-}\mathbf{XBF} \arrow[d, "\underset{s}{\boxtimes}"] & \\
& \mathcal{Z}(\mathcal{A})\text{-}\mathbf{XBF} \arrow[r, "\mathbf{DeEnrich}\overline{(-)}"] & \mathbf{BFC}/\mathcal{A},
\end{tikzcd}$$

where $\underset{s}{\boxtimes}$ was defined in Definition 4.7. This makes $\mathbf{BFC}/\mathcal{A}$ into a symmetric monoidal 2-category.

As outlined in the Introduction, Section 1.2.1, we want this product to be a braided fusion category containing $\mathcal{A}$, and to be computable in terms of just the tensor structure and braiding of $\mathcal{C}$ and $\mathcal{D}$. This is true by construction. The last requirement is that the unit is $\mathcal{Z}(\mathcal{A})$.

The Unit for the Reduced Tensor Product

Proposition 6.7. *Let $\mathcal{C} \in \mathbf{BFC}/\mathcal{A}$. Then we have an equivalence*

$$\mathcal{Z}(\mathcal{A}) \overset{\mathcal{A}}{\underset{\text{red}}{\boxtimes}} \mathcal{C} \cong \mathcal{C}.$$

Proof. Since $\mathbf{DeEnrich}\overline{(-)}$ and $\underleftarrow{(-)}$ are mutually inverse, this comes down to showing that $\underleftarrow{\mathcal{Z}(\mathcal{A})}$ is the unit for $\underset{s}{\boxtimes}$. To do this, observe that the unit for $\underset{s}{\boxtimes}$ is the free $\mathcal{Z}(\mathcal{A})_s$-enriched and tensored category on the category with one object and $\mathbb{I}_s$ as this object's endomorphism-object, and that $\underleftarrow{\mathcal{Z}(\mathcal{A})}$ is this category. $\square$

6.2.2 Basic Properties of the Reduced Tensor Product

Reduced Tensor Product and the Commutant of $\mathcal{A}$

It is interesting to examine what the reduced tensor product becomes on the commutant (Definition 5.23) of $\mathcal{A}$ in $\mathcal{C}$. When taking the reduced tensor product, this commutant behaves nicely. We will use the following bit of notation:

Notation 6.8. Let $\mathcal{C}$ and $\mathcal{D}$ be braided fusion categories containing $\mathcal{A}$. The symbol $\underset{\mathcal{A}}{\boxtimes}$, with slight abuse of notation, denotes

$$\mathcal{C} \underset{\mathcal{A}}{\boxtimes} \mathcal{D} = \mathbf{DeEnrich}(\underleftarrow{\mathcal{C}} \underset{\mathcal{A}}{\boxtimes} \underleftarrow{\mathcal{D}}),$$

where the use of $\underset{\mathcal{A}}{\boxtimes}$ on the right hand side denotes the $\mathcal{A}$-product introduced in Definition A.17.

Proposition 6.9. *Let $\mathcal{C}, \mathcal{D} \in \mathbf{BFC}/\mathcal{A}$. Then the commutant of $\mathcal{A}$ in $\mathcal{C} \overset{\mathcal{A}}{\underset{\text{red}}{\boxtimes}} \mathcal{D}$ satisfies:*

$$\mathcal{Z}_2(\mathcal{A}, \mathcal{C} \overset{\mathcal{A}}{\underset{\text{red}}{\boxtimes}} \mathcal{D}) \cong \mathcal{Z}_2(\mathcal{A}, \mathcal{C}) \underset{\mathcal{A}}{\boxtimes} \mathcal{Z}_2(\mathcal{A}, \mathcal{D}).$$

Proof. Using Proposition 5.24, this follows directly from Proposition 4.25. □

Examples

To give the reader some intuition for the reduced tensor product, we compute some examples.

Example 6.10. Let $\mathcal{C}$ be a braided fusion category containing a symmetric fusion category $\mathcal{A}$. Then the reduced tensor product over $\mathcal{A}$ of $\mathcal{C}$ with $\mathcal{A}$ is given by:

$$\mathcal{C} \overset{\mathcal{A}}{\underset{\text{red}}{\boxtimes}} \mathcal{A} \cong \mathcal{Z}_2(\mathcal{A}, \mathcal{C}) \underset{\mathcal{A}}{\boxtimes} \mathcal{A} \cong \mathcal{Z}_2(\mathcal{A}, \mathcal{C}).$$

To see this, we observe that the neutral subcategory of $\mathcal{A}$ enriched over itself is all of $\underleftarrow{\mathcal{A}}$. Now apply Proposition 4.25 to get:

$$\mathcal{C} \overset{\mathcal{A}}{\underset{\text{red}}{\boxtimes}} \mathcal{A} \cong \mathbf{DeEnrich}\,\overline{\underleftarrow{\mathcal{C}}_{\mathcal{A}} \underset{s}{\boxtimes} \underleftarrow{\mathcal{A}}} \cong \mathbf{DeEnrich}(\underleftarrow{\mathcal{Z}_2(\mathcal{A}, \mathcal{C})} \underset{\mathcal{A}}{\boxtimes} \underleftarrow{\mathcal{A}}) \cong \mathcal{Z}_2(\mathcal{A}, \mathcal{C}).$$

Here, we have used Corollary 5.25 and that $\underleftarrow{\mathcal{A}}$ is the unit for $\underset{\mathcal{A}}{\boxtimes}$.

Example 6.11. Let $\mathcal{C}$ and $\mathcal{D}$ be braided fusion categories containing $\mathcal{A}$, and assume that $\mathcal{D} = \mathcal{Z}_2(\mathcal{A}, \mathcal{D})$. Then:

$$\mathcal{C} \overset{\mathcal{A}}{\underset{\text{red}}{\boxtimes}} \mathcal{D} \cong \mathcal{Z}_2(\mathcal{A}, \mathcal{C}) \underset{\mathcal{A}}{\boxtimes} \mathcal{D}.$$

The assumption on $\mathcal{D}$ means that we have $\underleftarrow{\mathcal{D}}_{\mathcal{A}} = \underleftarrow{\mathcal{D}}$, by Proposition 5.24. Using Proposition 4.25 and Corollary 5.25, we get:

$$\mathcal{C} \overset{\mathcal{A}}{\underset{\text{red}}{\boxtimes}} \mathcal{D} \cong \mathbf{DeEnrich}\,\overline{\underleftarrow{\mathcal{C}}_{\mathcal{A}} \underset{s}{\boxtimes} \underleftarrow{\mathcal{D}}} \cong \mathbf{DeEnrich}(\underleftarrow{\mathcal{Z}_2(\mathcal{A}, \mathcal{C})} \underset{\mathcal{A}}{\boxtimes} \underleftarrow{\mathcal{C}}) \cong \mathcal{Z}_2(\mathcal{A}, \mathcal{C}) \underset{\mathcal{A}}{\boxtimes} \mathcal{D}.$$

Example 6.12. Let $\mathcal{C}$ and $\mathcal{D}$ be braided fusion categories containing $\mathcal{A}$, and assume that $\mathcal{C} = \mathcal{Z}_2(\mathcal{A}, \mathcal{C})$ and that $\mathcal{D} = \mathcal{Z}_2(\mathcal{A}, \mathcal{D})$. Then:

$$\mathcal{C} \overset{\mathcal{A}}{\underset{\text{red}}{\boxtimes}} \mathcal{D} \cong \mathcal{C} \underset{\mathcal{A}}{\boxtimes} \mathcal{D}.$$

The assumption on $\mathcal{C}$ and $\mathcal{D}$ means that we have $\underleftarrow{\mathcal{C}}_{\mathcal{A}} = \underleftarrow{\mathcal{C}}$ and $\underleftarrow{\mathcal{D}}_{\mathcal{A}} = \underleftarrow{\mathcal{D}}$, by Proposition 5.24. Using Proposition 4.25 and Corollary 5.25, we get:

$$\begin{aligned} \mathcal{C} \underset{\text{red}}{\overset{\mathcal{A}}{\boxtimes}} \mathcal{D} &\cong \mathbf{DeEnrich}\overline{\underleftarrow{\mathcal{C}} \underset{s}{\boxtimes} \underleftarrow{\mathcal{D}}} \\ &\cong \mathbf{DeEnrich}\,\underleftarrow{\mathcal{C}} \underset{\mathcal{A}}{\boxtimes} \underleftarrow{\mathcal{D}} \\ &\cong \mathcal{C} \underset{\mathcal{A}}{\boxtimes} \mathcal{D}. \end{aligned}$$

This in particular means that in this case computing the reduced tensor product does not require enriching over the Drinfeld centre.

6.3 Applications of the Reduced Tensor Product

6.3.1 Invertible Objects

Invertible Objects and Group Cohomology

We will now discuss the invertible objects for $\underset{\text{red}}{\overset{\mathcal{A}}{\boxtimes}}$ in the Tannakian case. The super-Tannakian case will be the subject of future work. We will make use of the classification of invertible G-crossed modular categories due to Turaev [Müg10]. This classification says that the group of invertible G-crossed modular categories is isomorphic to $H^3(G, U(1))$. The unit is given by $\mathbf{De} - \mathbf{Eq}(\mathcal{Z}(\mathcal{A}))$, where $\mathcal{A} = \mathrm{Rep}(G)$, and the isomorphism takes $\alpha \in H^3(G, U(1))$ to $\mathbf{De} - \mathbf{Eq}(\mathcal{Z}(\mathbf{Vect}_G^{\alpha}))$, the Drinfeld centre of the category of G-graded vector spaces with associator given by α. Because the equivalences in the diagram in Section 6.1.1 have either been shown to be symmetric monoidal or are symmetric monoidal by definition, this gives:

Proposition 6.13. *Let $\mathcal{A} = \mathrm{Rep}(G)$. The group of invertible objects in $\mathbf{BFC}/\mathcal{A}$ for $\underset{\text{red}}{\overset{\mathcal{A}}{\boxtimes}}$ is isomorphic to $H^3(G, U(1))$, with isomorphism given by $\alpha \mapsto \mathcal{Z}(\mathbf{Vect}_G^{\alpha})$.*

The author would be interested in finding a description of this group without reference to Tannaka duality.

6.3.2 Modular Categories

The Reduced Tensor Product of Modular Categories

To compare with known results, we have to examine what the reduced tensor product of two modular categories containing $\mathcal{A}$ is. For simplicity, we will assume $\mathcal{A}$ to be Tannakian. We will examine the super-case in future work. We will use the following:

Definition 6.14. A G-crossed braided fusion category $\mathcal{M}$ is called *modular* if $\mathcal{M}_e$ is a modular fusion category, and $\mathcal{M}_g$ is non-trivial for all $g \in G$.

The (de)-equivariantisation procedures take modular to modular.

Theorem 6.15 ([Müg10])**.** *If $\mathcal{M} \supset \mathrm{Rep}(G)$ is a modular tensor category, then* $\mathbf{De} - \mathbf{Eq}(\mathcal{M})$ *is a G-crossed modular category. Conversely,* $\mathbf{Eq}(\mathcal{N})$ *is a modular tensor category if $\mathcal{N}$ is a G-crossed modular category.*

Because the Deligne tensor product tensors modular categories to modular categories, the degree-wise tensor product of G-crossed modular categories is again G-crossed modular. This implies:

Proposition 6.16. *Let $\mathcal{C}$ and $\mathcal{D}$ be modular tensor categories containing a Tannakian subcategory $\mathcal{A}$. Then $\mathcal{C} \overset{\mathcal{A}}{\underset{\mathrm{red}}{\boxtimes}} \mathcal{D}$ is a modular tensor category.*

Minimal Modular Extensions

We will now compare the torsor structure for minimal modular extensions found in [LKW17b] with the reduced tensor product. Recall that a minimal modular extension is defined as follows.

Definition 6.17. Let $\mathcal{C}$ be a braided fusion category with Müger centre (Definition 5.23) $\mathcal{A}$. Then a *minimal modular extension* $\mathcal{M}$ of $\mathcal{C}$ is a modular tensor category such that

$$\mathcal{Z}_2(\mathcal{C}, \mathcal{M}) = \mathcal{A}, \tag{6.1}$$

where $\mathcal{Z}_2(\mathcal{C}, \mathcal{M})$ denotes the commutant of $\mathcal{C}$ in $\mathcal{M}$, see Definition 5.23. The set of minimal modular extensions of $\mathcal{C}$ will be denoted by $\mathrm{MME}(\mathcal{C})$.

It was conjectured in [Müg03b] that $\mathrm{MME}(\mathcal{C})$ is always non-empty. This conjecture was shown to be false in [Dria].

By the double commutant theorem [Müg03b, Theorem 3.2], the condition in Equation 6.1 above is equivalent to

$$\mathcal{Z}_2(\mathcal{A}, \mathcal{M}) = \mathcal{C}.$$

If $\mathcal{M}$ is a minimal modular extension of $\mathcal{C}$, we have

$$\underleftarrow{\mathcal{M}}_{\mathcal{A}} = \underleftarrow{\mathcal{C}}.$$

In particular, when $\mathcal{C} = \mathcal{A}$, we have

$$\underleftarrow{\mathcal{M}}_{\mathcal{A}} = \mathcal{A}_{\mathcal{Z}},$$

where $\mathcal{A}_{\mathcal{Z}} = \underleftarrow{\mathcal{A}}$ is the unit for $\underset{c}{\boxtimes}$, see Lemma 4.12. Combining Propositions 6.9 and 6.16 now gives:

Corollary 6.18. *Let $\mathcal{C}$ and $\mathcal{D}$ be braided fusion categories such that $\mathcal{Z}_2(\mathcal{C}) \cong \mathcal{Z}_2(\mathcal{D}) = \mathcal{A}$. Then the reduced tensor product on* $\mathbf{BFC}/\mathcal{A}$ *defines a pairing:*

$$- \underset{\text{red}}{\overset{\mathcal{A}}{\boxtimes}} - \colon \mathrm{MME}(\mathcal{C}) \times \mathrm{MME}(\mathcal{D}) \to \mathrm{MME}(\mathcal{C} \underset{\mathcal{A}}{\boxtimes} \mathcal{D}).$$

In [LKW17a], it was shown that $\mathrm{MME}(\mathcal{C})$, if non-empty, is a torsor for $\mathrm{MME}(\mathcal{A})$, and that $\mathrm{MME}(\mathcal{A}) \cong H^3(G, U(1))$ if $\mathcal{A} = \mathrm{Rep}(G)$. As $\mathcal{A}$ is the unit for $\underset{\mathcal{A}}{\boxtimes}$, our pairing in particular gives a pairing:

$$- \underset{\text{red}}{\overset{\mathcal{A}}{\boxtimes}} - \colon \mathrm{MME}(\mathcal{A}) \times \mathrm{MME}(\mathcal{A}) \to \mathrm{MME}(\mathcal{A}),$$

which by Proposition 6.13 makes $\mathrm{MME}(\mathcal{A})$ into a group isomorphic to $H^3(G, U(1))$. Furthermore, the pairing

$$- \underset{\text{red}}{\overset{\mathcal{A}}{\boxtimes}} - \colon \mathrm{MME}(\mathcal{A}) \times \mathrm{MME}(\mathcal{C}) \to \mathrm{MME}(\mathcal{C}),$$

gives an action of this group on $\mathrm{MME}(\mathcal{C})$.

Appendix A

Preliminaries

A.1 Enriched Category Theory

In this section we review some elements of enriched category theory. The contents of this section are widely known in the mathematical community.

A.1.1 Enriched tensored categories

Basics of enriched and tensored categories

Fix a spherical symmetric fusion category $\mathcal{A}$ with unit object $\mathbb{I}$ throughout. We assume the reader is familiar with the basic definition of a category enriched in $\mathcal{A}$. This section will deal with categories that are not only enriched, but also tensored over $\mathcal{A}$.

Notation A.1. The hom-objects in an $\mathcal{A}$-enriched category $\mathcal{C}$ between $c, c' \in \mathcal{C}$ will be denoted by $\underline{\mathcal{C}}(c, c')$. We will write $f\colon c \to_a c'$ for $f\colon a \to \underline{\mathcal{C}}(c, c')$. If $a = \mathbb{I}$, we will omit it from the notation. Furthermore, we will write $\mathcal{C}(c, c')$ for $\mathcal{A}(\mathbb{I}, \underline{\mathcal{C}}(c, c'))$.

We remind the reader of the following definition.

Definition A.2. Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be functors of $\mathcal{A}$-enriched categories. An *enriched natural transformation* from F to G is for each object $c \in \mathcal{C}$ a morphism $\eta_c\colon F(c) \to_{\mathbb{I}} G(c)$, that makes the following diagram commute for any $f\colon c \to_a c' \in \mathcal{C}$:

$$\begin{tikzcd} F(c) \arrow[r, "\eta_c"] \arrow[d, "F(f)", "a"'] & G(c) \arrow[d, "G(f)", "a"'] \\ F(c') \arrow[r, "\eta_{c'}"] & G(c'). \end{tikzcd}$$

Definition A.3. Let $\mathcal{C}$ be a category enriched in $\mathcal{A}$. Then $\mathcal{C}$ is called *tensored over* $\mathcal{A}$ if there exists, for every $c, c' \in \mathcal{C}$ and $a \in \mathcal{A}$ an object $a \cdot c$ together with a functorial isomorphism

$$\mathcal{A}\left(a, \underline{\mathcal{C}}(c, c')\right) \cong \mathcal{C}(a \cdot c, c'). \tag{A.1}$$

Definition A.3 allows us to write, denoting by $\mathcal{O}(\mathcal{A})$ a set of representatives for the isomorphism classes of simple objects in $\mathcal{A}$:

$$\underline{\mathcal{C}}(c, c') \cong \bigoplus_{a \in \mathcal{O}(\mathcal{A})} \mathcal{C}(ac, c')a. \tag{A.2}$$

This means we can view $f\colon c \to_a c'$ as a morphism $\tilde{f}\colon ac \to c'$, and the composite of $f\colon c \to_a c'$ and $g\colon c' \to_{a'} c''$ is given by

$$\tilde{g} \circ \left(\mathrm{id}_{a'} \cdot \tilde{f}\right) : a'ac \to c''.$$

Definition A.4. The images of morphisms under the isomorphism (A.1) are called *mates*. For $f\colon c \to_a c'$ we will write $\overline{f}\colon ac \to c'$ for its mate, and the mate of $g\colon ac \to c'$ will be denoted by $\underline{g}\colon c \to_a c'$.

Remark A.5. If $\mathcal{C}$ is enriched over $\mathcal{A}$, its category of enriched endofunctors $\mathrm{End}(\mathcal{C})$ is a tensor category, with the monoidal structure coming from composition. The assignment $a \mapsto a \cdot -$ extends to a functor $\mathcal{A} \to \mathrm{End}(\mathcal{C})$. This functor is in fact monoidal, cf. Lemma A.10.

Categories enriched and tensored over $\mathcal{A}$ form a 2-category, where we do have to take care functors between them are compatible with the tensor structure:

Definition A.6. The *2-category of $\mathcal{A}$-categories* $\mathcal{A}$**LinCat** is the 2-category where

- objects are categories enriched in and tensored over $\mathcal{A}$,
- morphisms $\mathcal{A}$-enriched functors $F\colon \mathcal{C} \to \mathcal{C}'$ equipped with a natural isomorphism

  $$F(a \cdot c) \xrightarrow[\cong]{\mu_{a,c}} a \cdot F(c),$$

  monoidal in a, such that the diagrams

  $$\begin{tikzcd} \mathcal{C}(a \cdot c, c') \arrow[rr, "\cong"] \arrow[d, "F"] & & \mathcal{A}(a, \mathcal{C}(c, c')) \arrow[d, "F"] \\ \mathcal{C}'(F(ac), F(c')) & \mathcal{C}'(a \cdot F(c), F(c')) \arrow[l, "\mu"'] & \mathcal{A}(a, \mathcal{C}'(c, c')), \arrow[l, "\cong"'] \end{tikzcd}$$

  commute for all $a \in \mathcal{A}$ and $c, c' \in \mathcal{C}$,
- and 2-morphisms enriched natural transformations $\eta\colon F \Rightarrow G$ that make the diagrams

$$\begin{tikzcd} F(ac) \arrow[r, "\eta_{ac}"] \arrow[d, "\cong"] & G(ac) \arrow[d, "\cong"] \\ a \cdot F(c) \arrow[r, "\mathrm{id}_a \cdot \eta_c"] & a \cdot G(c), \end{tikzcd}$$

commute for all $a \in \mathcal{A}$ and $c \in \mathcal{C}$.

Remark A.7. Definition A.6 is the most restrictive of the possible choices for a definition of $\mathcal{A}$**LinCat**. We could also have allowed that there is an auto-equivalence of $\mathcal{A}$ associated to every morphism between $\mathcal{A}$-enriched and tensored categories, and that each 2-morphism comes with a symmetric monoidal transformation between these auto-equivalences. However, Definition A.6 corresponds to the kind of enriched and tensored categories one obtains from module categories over $\mathcal{A}$.

Definition A.8. The *internal hom* between two objects $a, a' \in \mathcal{A}$ is the representing object $\underline{\mathcal{A}}(a, a')$ for the functor $a'' \mapsto \mathcal{A}(a''a, a')$. These hom objects make $\mathcal{A}$ into a category enriched and tensored over itself, i.e. a closed monoidal category.

Lemma A.9. *There is a canonical isomorphism*

$$\underline{\mathcal{A}}(\mathbb{I}, a) \cong a. \tag{A.3}$$

for all objects $a \in \mathcal{A}$.

Proof. Consider that

$$\mathcal{A}(a', \underline{\mathcal{A}}(\mathbb{I}, a)) \cong \mathcal{A}(a', a),$$

so $\mathcal{A}(\mathbb{I}, a)$ and a are canonically isomorphic under the Yoneda embedding. $\square$

The tensor structure of $\mathcal{C}$ over $\mathcal{A}$ induces an enriched natural isomorphism η with components

$$\eta_{a,c,c'} \colon \underline{\mathcal{A}}(a, \underline{\mathcal{C}}(c, c')) \to \underline{\mathcal{C}}(ac, c'). \tag{A.4}$$

To see this, observe that, given the natural transformation from (A.1), the definition of the enriched hom for $\mathcal{A}$ gives

$$\begin{aligned} \mathcal{A}(a', \underline{\mathcal{A}}(a, \underline{\mathcal{C}}(c, c'))) &\cong \mathcal{A}(a'a, \underline{\mathcal{C}}(c, c')) && \text{(A.5)} \\ &\cong \mathcal{A}(a', \underline{\mathcal{C}}(ac, c')) && \text{(A.6)} \end{aligned}$$

where the second line uses the isomorphism from (A.1). The preimage of this isomorphism under the Yoneda embedding is the desired isomorphism.

Lemma A.10. *Suppose $\mathcal{C}$ is tensored over $\mathcal{A}$. Then the functor $\mathcal{A} \to \mathrm{End}(\mathcal{C})$ taking a to $a \cdot -$ is a tensor functor.*

Proof. We only prove that there exist isomorphisms $a\cdot(a'\cdot c)\cong aa'\cdot c$ and omit checking the triangle and pentagon equations. We observe that

$$\begin{aligned}\mathcal{C}(a\cdot(a'\cdot c),c') &\cong \mathcal{A}(a,\underline{\mathcal{C}}(a'c,c'))\\ &\cong \mathcal{A}(a,\underline{\mathcal{A}}(a',\underline{\mathcal{C}}(c,c')))\\ &\cong \mathcal{A}(aa',\underline{\mathcal{C}}(c,c'))\\ &\cong \mathcal{C}(aa'c,c'),\end{aligned}\tag{A.7}$$

for all $c'\in\mathcal{C}$. □

Definition A.11. We let

$$\mathrm{ev}\colon \underline{\mathcal{C}}(c,c')\cdot c\to c'. \tag{A.8}$$

be the unit of the adjunction between $\underline{\mathcal{C}}(c,-)\colon \mathcal{C}\to\mathcal{A}$ and $-\cdot c\colon \mathcal{A}\to\mathcal{C}$ from (A.1).

Remark A.12. The evaluation morphism allows us to rewrite the defining diagram for an enriched natural transformation (Definition A.2) as follows:

$$\begin{tikzcd} \underline{\mathcal{C}}(c,c')\cdot F(c) \arrow[r, "\eta_c"] \arrow[d, "F\cdot\mathrm{id}"] & \underline{\mathcal{C}}(c,c')\cdot G(c) \arrow[d, "G\cdot\mathrm{id}"] \\ \underline{\mathcal{C}}(Fc,Fc')\cdot F(c) \arrow[d, "\mathrm{ev}"] & \underline{\mathcal{C}}(c,c')\cdot G(c) \arrow[d, "\mathrm{ev}"] \\ F(c') \arrow[r, "\eta_c"] & G(c'). \end{tikzcd}$$

Lemma A.13. *There exists a canonical isomorphism*

$$a\underline{\mathcal{C}}(c,c')\to\underline{\mathcal{C}}(c,ac').$$

Proof. We construct an isomorphism between the images of $a\underline{\mathcal{C}}(c,c')$ and $\underline{\mathcal{C}}(c,ac')$ under the Yoneda embedding:

$$\begin{aligned}\mathcal{A}(a',a\underline{\mathcal{C}}(c,c')) &\cong \mathcal{A}(a^*a',\underline{\mathcal{C}}(c,c'))\\ &\cong \mathcal{C}(a^*a'c,c')\\ &\cong \mathcal{C}(a'c,ac')\\ &\cong \mathcal{A}(a',\underline{\mathcal{C}}(c,ac')).\end{aligned}\tag{A.9}$$

□

Tensor product of enriched tensored categories

Definition A.14. The *enriched cartesian product* $\mathcal{C}\hat{\underset{\mathcal{A}}{\boxtimes}}\mathcal{D}$ of two $\mathcal{A}$-enriched categories $\mathcal{C}$ and $\mathcal{D}$ is the $\mathcal{A}$-enriched category whose objects are symbols $c\underset{\mathcal{A}}{\boxtimes}d$ with $c\in\mathcal{C}$ and $d\in\mathcal{D}$, and whose hom-objects are given by:

$$\underline{\mathcal{C}\hat{\underset{\mathcal{A}}{\boxtimes}}\mathcal{D}}\,(c\boxtimes d,c'\boxtimes d') :=\underline{\mathcal{C}}(c,c')\otimes\underline{\mathcal{D}}(d,d'), \tag{A.10}$$

where $\otimes$ is the tensor product in $\mathcal{A}$. Composition is given by first applying the braiding in $\mathcal{A}$ and then the compositions in $\mathcal{C}$ and $\mathcal{D}$.

Definition A.14 has an undesirable feature: if $\mathcal{C}$ and $\mathcal{D}$ from the above definition are semi-simple and idempotent complete, $\mathcal{C}\underset{\mathcal{A}}{\hat{\boxtimes}}\mathcal{D}$ in general will not be. Another, more prosaic, problem is that this notion of tensor product is not compatible with direct sums, we will fix this momentarily.

Definition A.15. The *Cauchy completion* of a $\mathcal{A}$-enriched category $\mathcal{C}$ is the category with objects n-tuples of objects from $\mathcal{C}$ together with a matrix of morphisms from $\mathcal{C}$ that is idempotent as morphism from the n-tuple to itself. Morphisms are matrices of morphisms from $\mathcal{C}$ that commute with the idempotents.

Remark A.16. Considering n-tuples of objects ensures compatibility with direct sums, picking idempotents ensures the category is idempotent complete. Note that $\mathcal{C}$ includes into its Cauchy completion. Any functor of $\mathcal{A}$-enriched categories induces a functor between the Cauchy completions, and the Cauchy completion of any category tensored over $\mathcal{A}$ is also tensored over $\mathcal{A}$.

Definition A.17. The $\mathcal{A}$-*product* $\mathcal{C}\underset{\mathcal{A}}{\boxtimes}\mathcal{D}$ of two $\mathcal{A}$-enriched categories $\mathcal{C}$ and $\mathcal{D}$ is the Cauchy completion of $\mathcal{C}\underset{\mathcal{A}}{\hat{\boxtimes}}\mathcal{D}$.

Proposition A.18. *If* $\mathcal{C},\mathcal{D}\in\mathcal{A}\mathbf{LinCat}$*, then* $\mathcal{C}\underset{\mathcal{A}}{\boxtimes}\mathcal{D}$ *is tensored over* $\mathcal{A}$ *with tensoring,*

$$a\cdot(c\boxtimes d)\cong(a\cdot c)\boxtimes d\cong c\boxtimes(a\cdot d), \tag{A.11}$$

and we have isomorphisms:

$$(a\cdot c)\boxtimes d\cong c\boxtimes(a\cdot d).$$

Proof. For the first part, recall, from Definition A.3, that it is enough to show that $c\boxtimes(a\cdot d)$ satisfies:

$$\underline{\mathcal{A}}(a,\underline{\mathcal{C}\underset{\mathcal{A}}{\boxtimes}\mathcal{D}}(c\boxtimes d,c'\boxtimes d'))\cong\underline{\mathcal{C}\underset{\mathcal{A}}{\boxtimes}\mathcal{D}}(c\boxtimes(ad),c'\boxtimes d'). \tag{A.12}$$

As $a\cdot(c\boxtimes d)$ is characterised by this equation, this will both establish existence of the tensor structure and $a\cdot(c\boxtimes d)\cong c\boxtimes(a\cdot d)$.

Substituting in the definition of the hom-objects in the $\mathcal{A}$-product, we see we are trying to find

$$\underline{\mathcal{A}}(a,\underline{\mathcal{C}}(c,c')\otimes\underline{\mathcal{D}}(d,d'))\cong\underline{\mathcal{C}}(c,c')\otimes\underline{\mathcal{D}}(ad,d').$$

Applying Lemma A.13 to $\mathcal{A}$ viewed as a category tensored over itself, we see that the left hand side reads:

$$\underline{\mathcal{C}}(c,c')\cdot\underline{\mathcal{A}}(a,\underline{\mathcal{D}}(d,d'))\cong\underline{\mathcal{C}}(c,c')\otimes\underline{\mathcal{D}}(ad,d'), \tag{A.13}$$

where the last isomorphism is (A.4). This gives us the desired isomorphism (A.12).

To establish the remaining assertion, observe that besides (A.13) we also have, after applying the symmetry in $\mathcal{A}$

$$\begin{aligned}\underline{\mathcal{A}}(a, \underline{\mathcal{C}}(c,c') \otimes \underline{\mathcal{D}}(d,d')) &\cong \underline{\mathcal{D}}(d,d') \cdot \underline{\mathcal{A}}(a, \underline{\mathcal{C}}(c,c')) \\ &\cong \underline{\mathcal{C}}(ac,c') \otimes \underline{\mathcal{D}}(d,d') \\ &\cong \underline{\mathcal{C} \hat{\boxtimes}_{\mathcal{A}} \mathcal{D}}((ac) \boxtimes_{\mathcal{A}} d, c' \boxtimes_{\mathcal{A}} d'),\end{aligned}$$

where we used the symmetry in $\mathcal{A}$ again the penultimate line. □

The $\mathcal{A}$-product is symmetric in the sense that:

Definition A.19. Let $\mathcal{C}, \mathcal{D} \in \mathcal{A}\mathbf{LinCat}$, then the *switch functor* $S\colon \mathcal{C} \boxtimes_{\mathcal{A}} \mathcal{D} \to \mathcal{D} \boxtimes_{\mathcal{A}} \mathcal{C}$ is defined by

$$c \boxtimes_{\mathcal{A}} d \mapsto d \boxtimes_{\mathcal{A}} c$$

at the level of objects and

$$\underline{\mathcal{C}}(c,c') \otimes \underline{\mathcal{D}}(d,d') \xrightarrow{s} \underline{\mathcal{D}}(d,d') \otimes \underline{\mathcal{C}}(c,c'),$$

where s is the symmetry in $\mathcal{A}$.

As the monoidal structure and the symmetry in $\mathcal{A}$ satisfy the appropriate coherence equations, the $\mathcal{A}$-product and the switch functor will strictly satisfy the coherence equations for a symmetric monoidal structure on the 2-category of categories enriched in and tensored over $\mathcal{A}$. That is, $(\mathcal{A}\mathbf{LinCat}, \boxtimes_{\mathcal{A}}, S)$ is a (strict) symmetric monoidal 2-category.

Given this $\mathcal{A}$-product, we can define:

Definition A.20. Let $\mathcal{C}$ be an $\mathcal{A}$-enriched category. Then a *$\mathcal{A}$-tensor structure* is a pair of functors:

$$\otimes\colon \mathcal{C} \boxtimes_{\mathcal{A}} \mathcal{C} \to \mathcal{C}, \qquad \mathbb{I}\colon \underline{\mathcal{A}} \to \mathcal{C},$$

equipped with associators and unitors satisfying the usual coherence conditions.

Proposition A.21. *The unit for the enriched cartesian product of enriched and tensored categories is $\mathcal{A}$ enriched over itself, denoted by $\underline{\mathcal{A}}$.*

Change of basis

Given monoidal categories $\mathcal{C}$ and $\mathcal{D}$, and a monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$, one gets a 2-functor $(F)_*$ from the 2-category of categories enriched over $\mathcal{C}$ to that of those enriched over $\mathcal{D}$, known as the "change of basis" functor. For a treatment of change of basis along monoidal functors, see [Cru08]. The proves from this reference translate straightforwardly to the lax monoidal case:

Definition A.22. A *lax monoidal functor* from a monoidal category $\mathcal{C}$ to a monoidal category $\mathcal{D}$ is a functor $F\colon \mathcal{C} \to \mathcal{D}$, together with a natural transformation:

$$\mu\colon F(-) \otimes F(-) \Rightarrow F(- \otimes -),$$

and a morphism

$$\mu^0\colon \mathbb{I}_{\mathcal{D}} \to F(\mathbb{I}_{\mathcal{C}}),$$

that satisfy the compatibility conditions with the associators $\alpha_{\mathcal{C}}$ and $\alpha_{\mathcal{D}}$:

$$\begin{tikzcd}
F(c)(F(c')F(c'')) \arrow[r, "\mu"] \arrow[d, "\alpha_{\mathcal{D}}"] & F(c)F(c'c'') \arrow[d, "\mu"] \\
(F(c)F(c'))F(c'') \arrow[d, "\mu"] & F(c(c'c'')) \arrow[dl, "F(\alpha_{\mathcal{C}})"] \\
F((cc')c'') &
\end{tikzcd},$$

for all $c, c', c'' \in \mathcal{C}$, and compatibility with the unitors:

$$\begin{tikzcd}
\mathbb{I}_{\mathcal{D}}F(c) \arrow[r, "\mu_0"] \arrow[d, "\lambda_{\mathcal{D}}"] & F(\mathbb{I}_{\mathcal{C}})F(c) \arrow[d, "\mu_{\mathbb{I},c}"] \\
F(c) & F(\mathbb{I}_{\mathcal{C}}c), \arrow[l, "F(\lambda_{\mathcal{C}})"]
\end{tikzcd}$$

and a similar condition for the right unitors.

In this section, we will focus on this lax case. We will make use of the following well-known results.

Proposition A.23. *Let $(F\colon \mathcal{C} \to \mathcal{D}, \mu)$ be a lax monoidal functor, and let $\mathcal{M}$ be a $\mathcal{C}$-enriched category. Then the category $F\mathcal{M}$ obtained from $\mathcal{M}$ by applying F to the hom-objects is a $\mathcal{D}$-enriched category, with composition given by the image of the composition in $\mathcal{M}$ under F and identity morphisms the image of the identity morphisms under F precomposed with μ^0.*

We will omit the proof of this statement, it is essentially the same as the proof of Lemma 4.9. It turns out that a change of basis is a 2-functor:

Proposition A.24. *Let $(F\colon \mathcal{C} \to \mathcal{D}, \mu)$ be a lax monoidal functor, then the assignment $\mathcal{M} \mapsto F\mathcal{M}$ extends to a 2-functor from the 2-category of $\mathcal{C}$-enriched categories to the 2-category of $\mathcal{D}$-enriched categories.*

If the monoidal categories involved are braided, we can additionally ask for the lax monoidal functor to be braided:

Definition A.25. Let $(F\colon \mathcal{C} \to \mathcal{D}, \mu)$ be a lax monoidal functor between braided (or symmetric) monoidal categories with braidings (or symmetries) $\beta^{\mathcal{C}}$ and $\beta^{\mathcal{D}}$, respectively. Then F is called *braided* (or *symmetric*) if the following diagram

$$\begin{tikzcd}
F(c)F(c') \arrow[r, "\beta^{\mathcal{D}}"] \arrow[d, "\mu"] & F(c')F(c) \arrow[d, "\mu"] \\
F(cc') \arrow[r, "F(\beta^{\mathcal{C}})"] & F(c'c)
\end{tikzcd}$$

commutes for all $c, c' \in \mathcal{C}$.

As discussed in the previous section, if the enriching category is symmetric monoidal, there is a notion of enriched Cartesian product (Definition A.14), and hence of enriched monoidal object. Change of basis along a symmetric lax monoidal functor preserves these monoidal objects. We give a proof as we will need a slight variation of this argument in this thesis. This fact is a consequence of the following.

Lemma A.26. *Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{C}$-enriched categories, where $(\mathcal{C}, s^{\mathcal{C}})$ is a symmetric monoidal category, and let $F\colon \mathcal{C} \to \mathcal{D}$ be a symmetric lax monoidal functor, and denote the symmetry in $\mathcal{D}$ by $s^{\mathcal{D}}$. Then the assignment:*

$$M\colon F\mathcal{M} \underset{\mathcal{D}}{\boxtimes} F\mathcal{N} \to F(\mathcal{M} \underset{\mathcal{C}}{\boxtimes} \mathcal{N}),$$

which is the idenity on objects and μ on hom-objects, is a $\mathcal{D}$-enriched functor.

Proof. We need to check that M preserves composition, this translates into checking that the outside of following diagram commutes:

$$\begin{tikzcd}
F\mathcal{M}F\mathcal{N}F\mathcal{M}F\mathcal{N} \arrow[r, "\mu\mu"] \arrow[d, "s^{\mathcal{D}}"] & F(\mathcal{M}\mathcal{N})F(\mathcal{M}\mathcal{N}) \arrow[d, "\mu"] \\
F\mathcal{M}F\mathcal{M}F\mathcal{N}F\mathcal{N} \arrow[d, "\mu\mu"] & F(\mathcal{M}\mathcal{N}\mathcal{M}\mathcal{N}) \arrow[d, "F(s^{\mathcal{D}})"] \\
F(\mathcal{M}\mathcal{M})F(\mathcal{N}\mathcal{N}) \arrow[r, "\mu"] \arrow[d, "F(\circ)F(\circ)"] & F(\mathcal{M}\mathcal{M}\mathcal{N}\mathcal{N}) \arrow[d, "F(\circ\circ)"] \\
F(\mathcal{M})F(\mathcal{N}) \arrow[r, "\mu"] & F(\mathcal{M}\mathcal{N}),
\end{tikzcd}$$

where we have suppressed the objects in for example $F\mathcal{M}(m, m')$ from the notation for readability. The bottom square commutes by naturality of μ, for the top square, we observe that the compatibility of μ with the associators (Definition A.22) allows us to rewrite this as:

$$\begin{tikzcd}
F\mathcal{M}F\mathcal{N}F\mathcal{M}F\mathcal{N} \arrow[r, "\mu"] \arrow[d, "s^{\mathcal{D}}"] & F\mathcal{M}F(\mathcal{N}\mathcal{M})F\mathcal{N} \arrow[r, "\mu\circ\mu"] \arrow[d, "F(s^{\mathcal{C}})"] & F(\mathcal{M}\mathcal{N}\mathcal{M}\mathcal{N}) \arrow[d, "F(s^{\mathcal{C}})"] \\
F\mathcal{M}F\mathcal{M}F\mathcal{N}F\mathcal{N} \arrow[r, "\mu"] & F\mathcal{M}F(\mathcal{M}\mathcal{N})F\mathcal{N} \arrow[r, "\mu\circ\mu"] & F(\mathcal{M}\mathcal{M}\mathcal{N}\mathcal{N}),
\end{tikzcd}$$

where the rightmost square commutes by naturality of μ, and the leftmost square is exactly the one from Defintion A.25. $\square$

We observe that when F is strong monoidal (so μ is an isomorphism), change of basis along F takes the $\mathcal{C}$-enriched Cartesian product to the $\mathcal{D}$-enriched Cartesian product.

Proposition A.27. *Let $F\colon \mathcal{C} \to \mathcal{D}$ and $\mathcal{M}$ be as in the previous lemma. Assume further that $\mathcal{C}$ and $\mathcal{D}$ are symmetric, and that F preserves the symmetry. Then, if $\mathcal{M}$ is $\mathcal{C}$-monoidal with monoidal structure $\otimes$, $F\mathcal{M}$ is $\mathcal{D}$ monoidal, with monoidal structure given by the composite*

$$F\mathcal{M} \underset{\mathcal{D}}{\boxtimes} F\mathcal{M} \xrightarrow{M} F(\mathcal{M} \underset{\mathcal{C}}{\boxtimes} \mathcal{M}) \xrightarrow{F(\otimes)} F\mathcal{M}.$$

Proof. The monoidal structure is clearly functorial, as it is a composite of $\mathcal{D}$-enriched functors. As μ respects the associators for $\mathcal{C}$, F will take the associators for $\mathcal{M}$ to associators for $F\mathcal{M}$, and similar for the unitors. $\square$

This extends to:

Proposition A.28. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a symmetric lax monoidal functor. Then the assignment $\mathcal{M} \mapsto F\mathcal{M}$ extends to a symmetric monoidal 2-functor from the 2-category of $\mathcal{C}$-enriched categories, with enriched cartesian product, to the 2-category of $\mathcal{D}$-enriched categories, with enriched cartesian product.*

In fact for a given $\mathcal{C}, \mathcal{D}$ monoidal categories and $\mathcal{C}$-enriched category $\mathcal{M}$, "change of basis" $(-)_*$ is itself a functor from the functor category $\mathbf{MonCat}^L(\mathcal{C}, \mathcal{D})$ of lax monoidal functors from $\mathcal{C}$ to $\mathcal{D}$ and their natural transformations to the category of $\mathcal{D}$-enriched categories. We remind the reader of the following definition:

Definition A.29. Let (F, μ) and (G, ν) be lax monoidal functors between $\mathcal{C}$ and $\mathcal{D}$, then a *lax monoidal natural transformation* $\sigma\colon F \Rightarrow G$ is a natural transformation such that for all $c, c' \in \mathcal{C}$ the following diagram commutes:

$$\begin{array}{ccc} F(c)F(c') & \xrightarrow{\mu_{c,c'}} & F(cc') \\ \big\downarrow{\scriptstyle \sigma_c\sigma_{c'}} & & \big\downarrow{\scriptstyle \sigma_{cc'}} \\ G(c)G(c') & \xrightarrow{\nu_{c,c'}} & G(cc'). \end{array}$$

Proposition A.30. *Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be lax monoidal functors and $\mathcal{M}$ be $\mathcal{C}$-enriched, and let $\sigma\colon F \Rightarrow G$ be a lax monoidal natural transformation. Then, for every $\mathcal{C}$-enriched category $\mathcal{M}$, we have a $\mathcal{D}$-enriched functor*

$$\Sigma\colon F\mathcal{M} \to G\mathcal{M},$$

given by the identity on objects and σ on the hom-objects. Furthermore, the assignment $\sigma \mapsto \Sigma$ preserves composition of natural transformations.

As being tensored is a property of the enrichment, the following is automatic.

Proposition A.31. *Let $\mathcal{M}$ be enriched and tensored over a monoidal category $\mathcal{C}$, and let $F\colon \mathcal{C} \to \mathcal{D}$ be a lax monoidal functor. Then $F\mathcal{M}$ is enriched and tensored over $\mathcal{D}$.*

A.2 Tensor Categories

A.2.1 Symmetric Fusion Categories

Tannaka Duality and Finite Super-Groups

A famous result by Deligne [Del90, Del02] states that every symmetric fusion category is the representation category of a (super-)group. Before we can state the Theorem, we need some definitions.

Definition A.32. Let $\mathcal{C}$ be a braided fusion category. A braided functor $\mathcal{C} \to$ **Vect** (or $\mathcal{C} \to$ **sVect**) is called a *(super-)fibre functor.* A braided fusion category $\mathcal{C}$ is called *(super-)Tannakian* if $\mathcal{C}$ admits a (super-)fibre functor.

Before we state Deligne's Theorem, we recall some basic facts about super-groups[1]:

Definition A.33. A *super-group* (G, ω) is a group G together with a choice of central element of order two ω. A *representation of a super-group* is a super-vector space V and a homomorphism $G \to \mathrm{Aut}_{\mathbf{sVect}}(V)$ that takes the element ω to the grading involution. The category of such representations $\mathrm{Rep}(G, \omega)$ is symmetric, with symmetry inherited from **sVect**. Observe that, as ω is central, an irreducible representation is homogeneous, and $\mathrm{Rep}(G, \omega)$ splits (as a linear category) as the sum of the subcategories of even representations and odd representations.

Lemma A.34. *The data of a supergroup (G, ω) is equivalent to the data of the quotient group $\overline{G} = G/\langle\omega\rangle$ together with a cocycle $\bar{\omega} \in H^2(\overline{G}, \mathbb{Z}_2)$.*

Proof. The quotient short exact sequence for $\overline{G}$ is a central extension of $\overline{G}$ by $\mathbb{Z}_2$, and such central extensions are classified by 2-cocycles with coefficients in $\mathbb{Z}_2$. □

Recall that, as a set, $G = \overline{G} \times \langle\omega\rangle$, with multiplication given by

$$(\bar{g}, \omega^n) \cdot (\bar{h}, \omega^m) = (\bar{g}\bar{h}, \omega^{n+m+\bar{\omega}(g,h)}).$$

Theorem A.35 ([Del90, Del02])**.** *Let $\mathcal{A}$ be a symmetric fusion category. Then $\mathcal{A}$ admits either a fibre functor or a super-fibre functor. Furthermore, the category $\mathcal{A}$ is equivalent as symmetric fusion category to the category of representations of the (super)-group of monoidal natural automorphisms of the (super)-fibre functor (where the grading involution is taken as the central order 2 element of the supergroup).*

A.2.2 The Drinfeld Centre

In this section we will remind the reader of some well-known facts about the Drinfeld centre.

[1] This definition of super-groups is different from viewing super-groups as group objects in the category of super-manifolds. The definition here is the one used in the context of fusion categories, see for example [BGH+17].

The Drinfeld Centre of a Monoidal Category

We recall the definition of the Drinfeld centre of a monoidal category for convenience.

Definition A.36. Let $\mathcal{M}$ be a monoidal category. The *Drinfeld centre* $\mathcal{Z}(\mathcal{M})$ *of* $\mathcal{M}$ is the braided monoidal category with objects pairs (m, β) where m is an object of $\mathcal{M}$ and β is a natural transformation

$$\beta\colon\ -\otimes m \Rightarrow m \otimes -.$$

The β are further required to satisfy

$$\beta_{nn'} = (\beta_n \otimes \mathrm{id}_{n'}) \circ (\mathrm{id}_n \otimes \beta_{n'}), \tag{A.14}$$

for all $n, n' \in \mathcal{M}$.

The morphisms in $\mathcal{Z}(\mathcal{M})$ are those morphisms in $\mathcal{M}$ that commute with the half-braidings in the obvious way. The tensor product is induced from the one on $\mathcal{M}$ and the braiding is the one specified by the half-braidings.

The Drinfeld centre comes with a *forgetful functor* $\Phi\colon \mathcal{Z}(\mathcal{A}) \to \mathcal{A}$, which forgets the half-braiding. This functor is monoidal, but not braided.

It is well known ([ENO05]) that the centre of a fusion category is again fusion. Furthermore, the centre of a spherical fusion category is a modular tensor category.

If $\mathcal{M}$ is braided, there is an obvious inclusion functor

$$\mathcal{M} \hookrightarrow \mathcal{Z}(\mathcal{M}), \tag{A.15}$$

which takes an object $m \in \mathcal{M}$ to $(m, \beta_{m,-})$, where $\beta_{m,-}$ denotes the natural isomorphism between $m \otimes -$ and $- \otimes m$ given by the braiding in $\mathcal{M}$.

A.2.3 The Drinfeld Centre of the Representation Category of a Finite Group

As discussed in Section A.2.1, every symmetric fusion category $\mathcal{A}$ is a representation category $\mathrm{Rep}(G)$ of a finite (super-)group. It turns out that the Drinfeld centre of a representation category of a finite group has the interesting feature that it is equivalent (as braided monoidal category) to the Drinfeld centre of the category of G-graded vector spaces, as we discuss in this section. We will first discuss the case of G being an ordinary finite group, then we move on to the supergroup case.

Tannakian Case

It is well-known ([BK01, Chapter 3.2]) that when $\mathcal{A} = \mathrm{Rep}(G)$, there is an equivalence:

$$\mathcal{E}\colon \mathcal{Z}(\mathcal{A}) \xrightarrow{\cong} \mathbf{Vect}_G[G], \tag{A.16}$$

between the Drinfeld centre and the category of equivariant vector bundles over G. This latter category is defined as follows:

Definition A.37. A *G-equivariant vector bundle on G* denoted by V is a collection of vector spaces V_g for $g \in G$, together with for each $h \in G$ isomorphisms

$$\rho_h \colon V_g \to V_{hgh^{-1}},$$

such that $\rho_{h'}\rho_h = \rho_{h'h}$. The vector space V_g will be called *the fibre over g*, and the isomorphisms ρ the *action data*.

Definition A.38. The *category of G-equivariant bundles on G*, $\mathbf{Vect}_G[G]$, is the category with objects G-equivariant bundles over G, and morphisms fibrewise linear maps that commute with the ρ_h.

Definition A.39. The *convolution tensor product* $V \otimes W$ of two equivariant bundles V, W over G is the bundle with fibres

$$(V \otimes W)_g = \bigoplus_{g_1 g_2 = g} V_{g_1} \otimes W_{g_2},$$

and action data $\rho_g = \oplus_{g_1 g_2 = g} \rho^V_{g_1} \otimes \rho^W_{g_2}$.

Furthermore, there is a braiding:

Definition A.40. The *braiding isomorphism*

$$\beta_{V,W} \colon V \otimes W \to W \otimes V$$

for $V, W \in \mathbf{Vect}_G[G]$, is given by using for each $g_1 g_2 = g$

$$V_{g_1} \otimes W_{g_2} \xrightarrow{s \circ (\mathrm{id} \otimes \rho_{g_1})} W_{g_1 g_2 g_1^{-1}} \otimes V_{g_1},$$

where s is the switch map of vector spaces, and summing this to a fibrewise map.

This makes $\mathbf{Vect}_G[G]$ into a braided monoidal category. It is in fact a modular fusion category, with simples supported by conjugacy classes of G. Note that, as the neutral element e is stabilised under conjugation by the whole group, the subcategory of vector bundles supported by the conjugacy class $[e]$ is the representation category of G. The inclusion functor from Equation (A.15) is in this model for the Drinfeld centre the functor that views a representation of G as a vector bundle over G supported by $[e]$.

Definition A.41. The *forgetful functor from $\mathbf{Vect}_G[G]$ to* $\mathrm{Rep}(G)$ is given by

$$\begin{aligned}\Phi \colon \mathbf{Vect}_G[G] &\to \mathrm{Rep}(G) \\ V = \{V_g\} &\mapsto \bigoplus_{g \in G} V_g,\end{aligned}$$

with G-action given by the action data.

Using the forgetful functor, the equivalence between $\mathcal{Z}(\mathrm{Rep}(G))$ and $\mathbf{Vect}_G[G]$ is in one direction given by taking $V = \{V_g\}$ and mapping it to $(\Phi(V), \beta_{V,-})$.

Definition A.42. Let G be a group. Then the category of G-graded vector spaces $\mathbf{Vect}[G]$ is the linear category with simple objects $\mathbb{C}$ of homogeneous degree g for $g \in G$, and fusion rules given by multiplication in the group.

The following is a straightforward computation:

Proposition A.43. *Let G be a finite group. Then the Drinfeld centre of* $\mathbf{Vect}[G]$ *is* $\mathbf{Vect}_G[G]$*.*

Super-Tannakian Case

We will now discuss the Drinfeld centre of the representation category of a finite supergroup (G, ω). We will denote the underlying finite group by G. We start with the following observation:

Lemma A.44. *For any finite supergroup (G, ω), there is an equivalence*

$$\mathcal{Z}(\mathrm{Rep}(G, \omega)) \cong \mathcal{Z}(\mathrm{Rep}(G))$$

of braided monoidal categories.

Proof. This follows directly from the fact that $\mathrm{Rep}(G, \omega)$ and $\mathrm{Rep}(G)$ are equivalent as monoidal categories. □

This means that the results from the previous section also apply to the super-Tannakian case, except for the following. Odd representations in $\mathrm{Rep}(G, \omega)$ braid along minus the identity with each other, so the inclusion functor $\mathrm{Rep}(G, \omega) \hookrightarrow \mathcal{Z}(\mathrm{Rep}(G, \omega))$ cannot be viewing these representation as bundles supported by $[e]$, these bundles braid trivially with each other. Instead, observe that, ω being central, the subcategory of bundles supported by $[\omega]$ is also the representation category of G (as linear category), but these bundles braid among each other according to the action of ω (see Definition A.40). In particular, odd representations will braid along minus the identity. In summary:

Proposition A.45. *Under the equivalence of $\mathcal{Z}(\mathrm{Rep}(G, \omega))$ with $\mathbf{Vect}_G[G]$, the inclusion functor*

$$\mathrm{Rep}(G, \omega) \hookrightarrow \mathbf{Vect}_G[G]$$

from Equation (A.15) *is given by viewing even and odd representations as vector bundles supported by $[e]$ and $[\omega]$, respectively.*

The forgetful functor to $\mathrm{Rep}(G, \omega)$ also differs compared to Definition A.41, we need to assign a parity to the images. To do this, it is helpful to observe the following:

Lemma A.46. *Let c be a simple object of $\mathbf{Vect}_G[G]$, then ω acts by either id_V or $-\mathrm{id}_V$.*

Proof. The simple objects in $\mathbf{Vect}_G[G]$ are supported by conjugacy classes. As ω is central, it has to act by the same linear map on each fibre. $\square$

With this Lemma in hand, we can simply define:

Definition A.47. Let c be a simple object in $\mathbf{Vect}_G[G]$, then c is called *even* (or *odd*) if ω acts as id (or $-\mathrm{id}$).

Now, the forgetful functor is again the functor to $\mathrm{Rep}(G, \omega)$ that takes the direct sum of the fibres, where we additionally remember the parity of the simple object it came from.

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