

The Riemann-Hilbert factorization problem in integrable systems

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Abstract

In this paper we give a mathematical formulation of the Riemann-Hilbert problem. We then proceed to show how the Riemann-Hilbert problem arises in the theory of integrable systems. To do this we formulate our problem in terms of loop algebras and outline some of their basic properties.

1 Introduction

In 1900, at the International Congress of Mathematicians, the famous mathematician David Hilbert put forth a list of what he perceived to be the most important unsolved problems in mathematics as a challenge for the mathematicians of the twentieth century to solve. Number twenty-one on this list was a rudimentary form of what are now known as Riemann-Hilbert problems.

The problem was, given initial data in the form of a representation of the fundamental group of the punctured Riemann-sphere, with the punctures located at given poles, can you always find an $N \times N$ linear system of differential equations such that:

$$\frac{d\Psi}{d\lambda} = A(\lambda)\Psi(\lambda), \quad (1)$$

with $A(\lambda)$ being an $N \times N$ matrix function depending rationally on the parameter λ , with poles at the given points, and with $\Psi(\lambda)$ generating the given representation. Such a system is called a Fuchsian system. Incidentally, it was found that, in general, such a system cannot be found [1]. There are, however, conditions on the initial data that guarantee the existence of a Fuchsian.

This question already reminds us of problems we face in constructing integrable models. There we start out from some analytic structure and ask ourselves whether this data represents an integrable system. The formulation of the Riemann-Hilbert factorization problem we are interested in is slightly different from the one above.

Because this problem is closely related to Riemann's idea that any function is completely determined by specifying its singularities and behaviour around these singularities, it got the name Riemann-Hilbert problem. While mathematicians strived to find a resolution of Hilbert's twenty-first problem, which was finally found to be a negative one by Bolibruch in 1989, the Riemann-Hilbert method was developed. For an overview of these developments, see [1]. This method

consists of reducing a problem to factorizing a given matrix-valued function defined on a curve into a part analytic inside and a part analytic outside the curve. These reductions to factorization problems also got the name Riemann-Hilbert problems. The original Riemann-Hilbert problem can be formulated in these terms and can thus be viewed as a special case of a Riemann-Hilbert problem.

The Riemann-Hilbert problem appears in many branches of mathematical physics. Apart from the application we will discuss here it has, for example, been shown [3] that dimensional regularization and renormalization in quantum field theory can be viewed as a Riemann-Hilbert problem. In the article [5] an overview of applications related to special functions appearing in integrable systems is given.

It should be noted that, even though all problems solved by following the above scheme carry the common name Riemann-Hilbert problem, the methods used for reducing these problems to a factorization problem are not described by a general algorithm. The reduction itself can be a quite non-trivial procedure.

In this paper we will first give a proof of the theorem on Riemann-Hilbert factorization that is relevant for our purposes. Then we briefly discuss loop groups and loop algebras, since the factorization arises quite naturally in this setting for integrable systems. Then we show how the factorization can be used to solve equations in integrable systems with a single pole Lax matrix, and generalize this to the case of multiple poles.

2 Riemann-Hilbert factorization

In this section we will discuss the theorem on the particular form of the Riemann-Hilbert problem we require. The theorem is best formulated on the one-point compactification of the complex plane, the Riemann sphere, denoted $\bar{\mathbb{C}}$. (One can visualize this by imagining a sphere resting with its south pole on the origin of the plane, associating to each point on the plane the point of intersection with the sphere of the line between the point and the north pole. The north pole then represents the added point at infinity.) We will limit ourselves to the existence part of the proof, since this provides some insight into the structure of the construction.

Theorem 2.1. *Let C be the unit circle in the Riemann sphere $\bar{\mathbb{C}}$, let $g : C \rightarrow \text{Mat}(\bar{\mathbb{C}}, n)$ be a given analytic matrix-valued function on this circle, with $\det(g) \neq 0$ on C . Let η be such that g has an analytic extension to the annulus $\{\lambda \in \bar{\mathbb{C}} : 1 - \eta < |\lambda| < 1 + \eta\}$. Let $U_+ = \{\lambda \in \bar{\mathbb{C}} : |\lambda| < 1 + \eta\}$ and $U_- = \{\lambda \in \bar{\mathbb{C}} : |\lambda| > 1 - \eta\}$. Then there exist analytic $g_{\pm} : U_{\pm} \rightarrow \text{GL}(\bar{\mathbb{C}}, n)$ and $\Lambda(\lambda) = \text{diag}(\lambda^{k_1}, \dots, \lambda^{k_i}, \dots, \lambda^{k_n})$, with $k_i \in \mathbb{Z}$ unique up to ordering, such that, on the annulus $U_+ \cap U_-$:*

$$g(\lambda) = g_-^{-1}(\lambda)\Lambda(\lambda)g_+(\lambda). \quad (2)$$

Furthermore, these g_{\pm} are unique if we require $\lim_{\lambda \rightarrow \infty} g_-(\lambda) = \text{Id}$.

Proof. We start by constructing a suitable candidate for $g_+(\lambda)$. To do this we consider the operator on matrix-valued continuous functions on C given by:

$$(\mathcal{F}f)(\lambda) = f(\lambda) + \frac{1}{2\pi i} \oint_C \frac{K(\lambda, \lambda')}{\lambda - \lambda'} f(\lambda') d\lambda', \quad (3)$$

where the kernel $K(\lambda, \lambda')$ is analytic on C and depends on $g(\lambda)$:

$$K(\lambda, \lambda') = g^{-1}(\lambda)g(\lambda') - 1. \quad (4)$$

This is a so-called Fredholm operator, since it is the sum of an invertible and a compact operator. One of the basic properties of Fredholm operators is (see for example [4]) that its image is closed and has a finite dimensional complement. Since the matrices of polynomials are dense in the space of matrices of continuous functions we can always find a matrix of polynomials $P(\lambda)$, with $\det P(\lambda) \neq 0$ for large enough λ , such that $g^{-1}(\lambda)P(\lambda)$ is in the image of \mathcal{F} . This means we can always find a matrix-valued function f such that, for such a $P(\lambda)$:

$$(\mathcal{F}f)(\lambda) = g^{-1}(\lambda)P(\lambda). \quad (5)$$

We can extend this function analytically to the annulus $U_+ \cap U_-$, since both g and P have extensions, combining (3) and (5) we get:

$$f(\lambda) = g^{-1}(\lambda)P(\lambda) + \frac{1}{2\pi i} \oint_C \frac{K(\lambda, \lambda')}{\lambda - \lambda'} f(\lambda') d\lambda'. \quad (6)$$

Any function analytic on an annulus has a Laurent series. This means we can write $f(\lambda) = f(\lambda)_- + f_+(\lambda)$, with f_{\pm} analytic on U_{\pm} and $f_-(\infty) = 0$. The singular part can be extracted using a contour integral:

$$f_-(\lambda) = -\frac{1}{2\pi i} \oint_C \frac{f(\lambda')}{\lambda' - \lambda} d\lambda'. \quad (7)$$

The regular part $f_+(\lambda)$ can now be obtained by simply subtracting $f_-(\lambda)$ from the expression for the analytic extension to the annulus:

$$f_+(\lambda) = g^{-1}(\lambda)P(\lambda) + \frac{1}{2\pi i} \oint_C \frac{g^{-1}(\lambda)g(\lambda')f(\lambda')}{\lambda - \lambda'} d\lambda'. \quad (8)$$

Multiplying this from the left by $g(\lambda)$ we get an equation of the form (2):

$$g(\lambda)f_+(\lambda) = P(\lambda) + \frac{1}{2\pi i} \oint_C \frac{g(\lambda')f(\lambda')}{\lambda - \lambda'} d\lambda' \equiv \tilde{f}_-, \quad (9)$$

where we still need to find the exact expressions for $g_{\pm}(\lambda)$ and $\Lambda(\lambda)$. To satisfy the conditions in the theorem $g_{\pm}(\lambda)$ should have non-vanishing determinants on their respective domains and $g_-(\lambda)$ should be made to tend to the identity as λ tends to infinity. The analyticity of g_{\pm} can be derived from the analyticity of f_+ and f_- . By construction, $f_+(\lambda)$ is analytic on U_+ , but it can still have a zero determinant at some points. The function $\tilde{f}_-(\lambda)$ is, also by construction, analytic on U_- , and behaves as $P(\lambda)$ for large $|\lambda|$, since for such λ the integral term is suppressed. So for large $|\lambda|$ the determinant of $\tilde{f}_-(\lambda)$ is non-zero. This means that $\det \tilde{f}_-(\lambda)$ and $\det f_+(\lambda)$ do not vanish identically and it is a basic fact from complex analysis that therefore these functions have a finite number of zeroes at finite distance.

We now proceed to remove these zeroes. Suppose $f_+(\lambda)$ has zero determinant at λ_0 , and that this zero is simple. This implies that we can find a matrix M such that $f_+(\lambda_0)M$ has vanishing first column. Since the zero is simple, multiplying from the right by $\text{diag}(1/(\lambda - \lambda_0), 1, \dots, 1)$ will remove it. Viewing degenerate zeroes as products of simple zeroes we can iterate this procedure to remove all zeroes from $\det f_+(\lambda)$. To make sure (9) is still satisfied all multiplication from

the right should of course also be performed on $\tilde{f}_-(\lambda)$. This, however, does not spoil the analyticity of $\tilde{f}_-(\lambda)$ on U_- , the zeroes of $\det f_+(\lambda)$ are all contained in $U_+ - U_-$. Conversely, applying the same procedure to $\tilde{f}_-(\lambda)$ will keep the analytic properties of $f_+(\lambda)$ intact. For large $|\lambda|$ we can approximate

$$\frac{1}{\lambda - \lambda_0} = \frac{1}{\lambda} \left(1 + \mathcal{O}\left(\frac{\lambda_0}{\lambda}\right) \right), \quad (10)$$

and we get a matrix $f_-(\lambda)$ behaving at ∞ as:

$$f_- = g_-^{-1}(\lambda) \text{diag}(\lambda^{k_1}, \dots, \lambda^{k_N}) \quad \det g_-(\lambda) \neq 0 \quad (11)$$

This is the defining equation for $g_-(\lambda)$. Finally, setting g_+ equal to our modified f_+ and $\Lambda(\lambda) = \text{diag}(\lambda^{k_1}, \dots, \lambda^{k_N})$, we have obtained the factorization (2). \square

Example 2.1. This proof also yields an algorithm for determining the indices k_i for $g(\lambda) = 1 + \epsilon h(\lambda)$, with ϵ sufficiently small. Up to linear order in ϵ we have $g^{-1}(\lambda) = 1 - \epsilon h(\lambda) + \mathcal{O}(\epsilon^2)$, and $K(\lambda, \lambda')$ simplifies to:

$$K(\lambda, \lambda') = \epsilon(h(\lambda') - h(\lambda)) + \mathcal{O}(\epsilon^2). \quad (12)$$

For ϵ small enough, this means that \mathcal{F} is surjective, for any function $f(\lambda)$ we have that

$$\tilde{f}(\lambda) = f(\lambda) - \frac{1}{2\pi i} \oint_C \frac{K(\lambda, \lambda')}{\lambda - \lambda'} f(\lambda') d\lambda' \quad (13)$$

satisfies $(\mathcal{F}\tilde{f})(\lambda) = f(\lambda) + \mathcal{O}(\epsilon^2)$. Because \mathcal{F} is surjective, $g^{-1}(\lambda)$ is in its image and we can pick $P(\lambda) = 1$. We then get for the analytic extension (6) of our $f(\lambda)$:

$$f(\lambda) = 1 - \epsilon \left(h(\lambda) - \frac{1}{2\pi i} \oint_C \frac{h(\lambda') - h(\lambda)f(\lambda')}{\lambda - \lambda'} \right). \quad (14)$$

So the polar parts are of order ϵ or smaller, and we can decompose $f_+(\lambda) = 1 + \mathcal{O}(\epsilon)$, and we get from (9) that $\tilde{f}_-(\lambda) = 1 + \mathcal{O}(\epsilon)$. So both $f_+(\lambda)$ and $\tilde{f}_-(\lambda)$ have non-vanishing determinants and all k_i vanish.

3 Loop algebras

To see how the Riemann-Hilbert problem arises naturally in integrable systems it is convenient to view the Lax equation from the point of so-called loop algebras. We will assume the reader is familiar with the basic concepts of Lie groups, algebras and their duals, the adjoint action and its counterpart in the dual the coadjoint action. For an explanation of these concepts we refer the reader to for example the treatment in [2].

Definition 3.1. A loop group \mathcal{G} over a Lie group G is the group of all analytic G -valued functions on the interior of a loop $C \subset \mathbb{C}$, with pointwise (in the C -variable) multiplication.

For clarity of exposition we will take $C = S^1$ in what follows, and assume G to be a matrix group. Expanding an element $g(\lambda) \in \mathcal{G}$ as a formal power series, we see that $g(\lambda) = \sum_{r=0}^{\infty} g_r \lambda^r$. As usual, the Lie algebra \mathfrak{g} of \mathcal{G} is defined as the tangent space to the identity element and consists of elements of the form $\sum_{r=0}^{\infty} X_r \lambda^r$, with pointwise commutator. This Lie algebra is called a loop algebra.

The real reason these loop algebras are of interest to us lies in the form the elements of the dual take. For functions on a loop it is quite natural to define an inner product by:

$$\langle f, g \rangle = \frac{1}{2\pi i} \oint_C f(\lambda) g(\lambda) d\lambda. \quad (15)$$

On matrix-spaces a nice choice for an inner product is:

$$\langle \Xi, X \rangle = \text{Tr}(\Xi X). \quad (16)$$

Putting these together allows us to find the elements of the dual \mathfrak{g}^* . They should be functions $\Xi(\lambda)$ on C which have a bounded pairing with elements $X \in \mathfrak{g}$. The pairing is:

$$\langle \Xi(\lambda), X(\lambda) \rangle = \text{Tr} \left(\frac{1}{2\pi i} \oint_C \Xi(\lambda) X(\lambda) d\lambda \right) = \text{Tr} \text{Res}_{\lambda=0}(\Xi(\lambda) X(\lambda)). \quad (17)$$

From this we see that in order to get a non-zero pairing, $\Xi(\lambda)$ should have negative powers of λ . To make sure the pairing is bounded for all $X(\lambda) \in \mathfrak{g}$ we should take series with only a finite number of terms with negative powers in λ . So elements $\Xi(\lambda) \in \mathfrak{g}^*$ are of the form $\Xi(\lambda) = \sum_{r=1}^{\infty} \Xi_r \lambda^{-r}$ with only a finite number of Ξ_r non-zero. The pairing between elements $\Xi(\lambda) \in \mathfrak{g}^*$ and $X(\lambda) \in \mathfrak{g}$ then takes the simple form:

$$\langle \Xi(\lambda), X(\lambda) \rangle = \sum_{r=0}^{\infty} \Xi_{r+1} X_r. \quad (18)$$

Recall that when developing the Lax formalism with spectral parameter (see also [2]), one finds that the Lax matrix can be diagonalized around each pole λ_k :

$$L(\lambda) = g^{(k)}(\lambda) A^{(k)}(\lambda) g^{(k)-1}(\lambda), \quad (19)$$

with $g^{(k)}(\lambda)$ a matrix regular around λ_k and $A^{(k)}(\lambda)$ a diagonal matrix. One can thus decompose:

$$L(\lambda) = L_0 + \sum_k L_k(\lambda) \quad (20)$$

with the L_k given by:

$$L_k(\lambda) = \left(g^{(k)}(\lambda) A^{(k)}(\lambda) g^{(k)-1}(\lambda) \right)_-, \quad (21)$$

where the subscript minus denotes taking the polar part at the relevant pole λ_k .

So for a Lax matrix with a single pole at 0, we can interpret the Lax matrix as an element of the dual \mathfrak{g}^* . The regular matrices $g^{(k)}(\lambda)$ can be interpreted as being elements of \mathcal{G} acting on \mathfrak{g}^* with the coadjoint action:

$$\text{Ad}^*(g)\Xi(X) = \Xi(\text{Ad}(g^{-1})) = \Xi(g^{-1}Xg), \quad (22)$$

so with our pairing:

$$\text{Ad}^*(g)\Xi = g\Xi g^{-1}. \quad (23)$$

This fits in nicely with the diagonalization scheme encountered when developing the Lax formalism, with the matrix M playing the role of a connection, with transformation law:

$$M_g = gMg^{-1} + \dot{g}g^{-1}, \quad (24)$$

where the overdot denotes the time-derivative. The Lax equation is then just the covariant derivative of $L(\lambda, t)$ set to zero:

$$0 = \nabla_M L = \dot{L} - [M, L]. \quad (25)$$

So we see that, for a time-independent $A(\lambda)$, the time evolution of the Lax matrix is a curve through a coadjoint orbit determined by $A(\lambda)$.

The loop algebra formalism also yields a nice scheme for constructing a Poisson bracket on the matrix elements of $L(\lambda)$ which makes sure the eigenvalues are in involution. This Poisson bracket is called the Konstant-Kirillov bracket. This construction is unfortunately beyond the scope of this paper. For more information, see [2].

In the case of multiple poles one constructs a loop algebra \mathfrak{g}_k around each pole λ_k by considering the maps from a small loop around λ_k (and only enclosing one pole) to the Lie group G . Taking the direct sum of these algebras we get a loop algebra \mathfrak{g} . This algebra has associated to it a Lie group which is the direct product of loop groups \mathcal{G}_k consisting of matrices regular around λ_k .

4 Factorization in integrable systems

In the scheme presented above M is chosen such that it satisfies the transformation law (24) for all $g \in \mathcal{G}$. In the construction of integrable system one encounters the converse question: given a connection M , which $g(t) \in \mathcal{G}$ satisfy (24) and the Lax equation? The Lax equation makes sure that $L(\lambda)$ and $M(\lambda)$ are simultaneously diagonalized, if $A^{(k)}(\lambda) = g^{(k)-1}(\lambda)L(\lambda)g^{(k)}(\lambda)$ is a diagonalization for $L(\lambda)$ around λ_k , then the correspondingly transformed $M(\lambda)$:

$$B^{(k)}(\lambda) = g^{(k)-1}(\lambda)M(\lambda)g^{(k)}(\lambda) - g^{(k)-1}(\lambda)\dot{g}^{(k)}(\lambda) \quad (26)$$

should be diagonal to make sure the right hand side of

$$\dot{A}^{(k)}(\lambda) = \left[B^{(k)}(\lambda), A^{(k)}(\lambda) \right] \quad (27)$$

is diagonal. We can view (26) as an equation of motion for $g(\lambda, t)$. In order to do a neat analysis of this equation of motion it is convenient to pick a specific form for $M(\lambda)$. From matching the poles on both sides of (27) one finds that the most general form for $M(\lambda)$ is:

$$M(\lambda) = M_0 + \sum_k M_k, \quad \text{with} \quad M_k(\lambda) = \left(g^{(k)}(\lambda)B^{(k)}(\lambda)g^{(k)-1}(\lambda) \right)_-. \quad (28)$$

This means we can analyse all possible choices for $M(\lambda)$ by looking at all possible choices for $B^{(k)}(\lambda)$. One has three properties to fix when constructing $B^{(k)}(\lambda)$:

which diagonal entries are non-zero, where in the λ -plane they have their poles and what the order of these poles is. This means we can span the space of possible $B^{(k)}(\lambda)$ by diagonal matrices with a single pole of order n at λ_k in the $\alpha\alpha$ entry. That is, matrices ξ_i with $i = (k, n, \alpha)$ given by:

$$\xi_i(\lambda) = \frac{E_{\alpha\alpha}}{(\lambda - \lambda_k)^n}, \quad (29)$$

where $E_{\alpha\alpha}$ denotes the matrix with a one in entry $\alpha\alpha$ and the rest of the entries zero. This leads us to the consideration of M_i :

$$M_i = \left(g^{(k)} \xi_i g^{(k)-1} \right)_k, \quad (30)$$

where the subscript k denotes taking the polar part at λ_k , and the t_i -time evolutions of L associated with them through the Lax equation:

$$\partial_{t_i} L = [M_i, L], \quad (31)$$

are called the elementary flows. It turns out that these flows commute, and decomposing a given M into M_i corresponds to decomposing its flow into a composition of elementary ones.

Plugging these M_i in to (26), setting $i = (k, n, \alpha)$ and solving for $\partial_{t_i} g^{(k')}$ yields the equation of motion:

$$\partial_{t_i} g^{(k')} = M_i g^{(k')} - g^{(k')} B_i^{(k')}. \quad (32)$$

Here $B_i^{(k')}$ is M_i diagonalized around $\lambda_{k'}$. Now for $k = k'$ we have $B_i^{(k')} = \xi_i + \text{regular}$, since $g^{(k)}$ is regular around λ_k and the only polar contribution comes from M_i . For $k \neq k'$ all other terms in the equation are regular around $\lambda_{k'}$, it follows that $B_i^{(k')}$ is regular around $\lambda_{k'}$. Note that the requirement that $g^{(k')}(\lambda)$ diagonalizes $L(\lambda)$ does not fix $g^{(k')}(\lambda)$ completely. Multiplication from the right by a regular diagonal matrix $d^{(k')}(\lambda)$ will not change the diagonalization. We can thus redefine $g^{(k')}(\lambda) \mapsto g^{(k')}(\lambda) d^{(k')}(\lambda)$, this changes (32) and absorbing this change into $B_i^{(k')}$ we get $B_i^{(k')}(\lambda) \mapsto B_i^{(k')}(\lambda) + B_i^{(k')^{-1}}(\lambda) \partial_{t_i} d^{(k')}(\lambda)$. This freedom can be used to remove the regular part of $B_i^{(k')}(\lambda)$, for details see [2]. We have thus found for our equations of motion that for $k = k'$:

$$\partial_{t_i} g^{(k)} = \left(M_i - g^{(k)} \xi_i g^{(k)-1} \right) g^{(k)} = - \left(g^{(k)} \xi_i g^{(k)-1} \right)_+ g^{(k)}, \quad (33)$$

and for $k \neq k'$:

$$\partial_{t_i} g^{(k')} = M_i g^{(k')}. \quad (34)$$

We now proceed to reduce (33) to a factorization problem. In order to make the notation less cumbersome, we treat the single pole case with the pole at $\lambda_k = 0$ first. For clarity we also relabel $g^{(k)}(\lambda) = g_+(\lambda)$. In this notation we have:

$$\partial_{t_i} g_+ = - \left(g_+ \xi_i g_+^{-1} \right)_+ g_+ \quad (35)$$

as our equations of motion, one for each $i = (n, \alpha)$. We claim that (35) can be solved by solving the factorization problem, in the notation of theorem 2.1:

$$g_-^{-1}(\lambda, t) g_+(\lambda, t) = e^{\sum_j \xi_j t_j} g_+(\lambda, 0) e^{-\sum_j \xi_j t_j}, \quad (36)$$

where $t = (t_{1,1}, t_{1,2}, \dots)$. It is easy to see that the $g_+(\lambda, t)$ obtained from solving the factorization does indeed solve (35). Take the derivative with respect to t_i of (36), and split into $g_+(\lambda, t)$ and $g_-(\lambda, t)$ terms by multiplying from the left by $g_-(\lambda, t)$ and from the right by $g_+^{-1}(\lambda, t)$. This yields (suppressing the (λ, t) -dependence):

$$-\partial_{t_i} g_- g_-^{-1} + \partial_{t_i} g_+ g_+^{-1} = g_- \xi_i g_-^{-1} - g_+ \xi_i g_+^{-1}. \quad (37)$$

Our usual equating the polar parts of the equation now gives:

$$\partial_{t_i} g_+(\lambda, t) g_+^{-1}(\lambda, t) = - (g_+(\lambda, t) \xi_i(\lambda) g_+^{-1}(\lambda, t))_+, \quad (38)$$

$$\partial_{t_i} g_-(\lambda, t) g_-^{-1}(\lambda, t) = -g_-(\lambda, t) \xi_i(\lambda) g_-^{-1}(\lambda, t) + (g_+(\lambda, t) \xi_i(\lambda) g_+^{-1}(\lambda, t))_- \quad (39)$$

where we used that $g_-(\lambda, t) \xi_i g_-^{-1}(\lambda, t)$ is purely singular around zero. We see that the first of these equations is (35) and we conclude that solving the factorization problem automatically solves our equations of motion.

The question remains why this scheme works, it is rather surprising that we can rewrite a system of differential equations in terms of an algebraic problem. Recall that our matrices $g_+(\lambda, t)$ can be seen as elements of some loop group \mathcal{G} , and our solution should be a curve through this group. In our factorization the matrices $g_-(\lambda, t)$ occur. Since these are purely singular inside our loop, we can interpret them as being in the image under the exponential map of \mathfrak{g}^* . This implies that to properly interpret (36) we have to consider the full loop algebra:

$$\tilde{\mathfrak{g}} = \mathfrak{g} + \mathfrak{g}^*, \quad (40)$$

with elements of the form $\sum_{r=-n}^{\infty} X_r \lambda^r$ and the usual pointwise matrix commutator. Any element $X \in \tilde{\mathfrak{g}}$ can be decomposed into a polar and a singular part, this corresponds to decomposing

$$X = X_+ - X_-, \quad (41)$$

with $X_+ \in \mathfrak{g}$ and $X_- \in \mathfrak{g}^*$. The minus sign is conventional and chosen to match the formulation of the factorization problem, at the group level we have the decomposition:

$$g(\lambda) = g_-^{-1}(\lambda) g_+(\lambda), \quad (42)$$

with $g_+(\lambda) \in \exp(\mathfrak{g})$ and $g_-(\lambda) \in \exp(\mathfrak{g}^*)$. This means that the decomposition (41) corresponds to a Riemann-Hilbert problem. So what should the formulation of this Riemann-Hilbert problem be in our case? Equation (35) tells us we are looking for $g_+(\lambda) \in \exp(\mathfrak{g})$ and we have the boundary condition that $g_+(\lambda, 0)$ is regular around $\lambda = 0$ and diagonalizes $L(\lambda, 0)$. Looking at (36) we see that we have $g(\lambda, t)$ equal to the right hand side there. This expression is that of the representation of $g(\lambda, 0)$ at time t under the flow of $\xi_i(\lambda)$. This makes sense, in the full loop algebra $\xi_i(\lambda)$ generates the same time evolution as the Lax-equation. This simple evolution does, however, not take into account the regularity conditions. To find the regular part $g_+(\lambda, t)$ we have to solve the factorization (36).

This leads us to a reformulation that is more convenient for the multipole case, define:

$$\theta_-^{-1}(\lambda, t) = e^{-\sum_i \xi_i(\lambda) t_i} g_-^{-1}(\lambda, t), \quad \theta_+(\lambda, t) = g_+(\lambda, t) g_+(\lambda, 0). \quad (43)$$

Our factorization problem then becomes:

$$\theta_-^{-1}(\lambda, t)\theta_+(\lambda, t) = \exp\left(-g_+(\lambda, 0) \sum_i \xi_i(\lambda) t_i g_+^{-1}(\lambda, 0)\right), \quad (44)$$

where the right hand side is the flow of $\xi(\lambda)$ in the frame where $L(\lambda, 0)$ is diagonal. For the case of multiple poles we now construct a loop algebra around each point, and take the direct sum to get the loop algebra for our problem. We can now formulate the Riemann-Hilbert problem in terms of elements $\tilde{g}^{(k)}(\lambda)$ defined on loops $\Gamma^{(k)}$ around the poles λ_k , and look for matrices $g_+(\lambda)$ and $g_-(\lambda)$, analytic inside respectively outside all $\Gamma^{(k)}$, satisfying:

$$g_-^{-1}(\lambda)g_+(\lambda) = \tilde{g}^{(k)}(\lambda), \quad \lambda \in \Gamma^{(k)}, \quad (45)$$

for all k . The solution to this Riemann-Hilbert can be found simply by first solving the problem on the first contour $\Gamma^{(1)}$. Call this solution $g_{\pm}^{(1)}(\lambda)$. We can now look for the full solution $g_{\pm} = f g_{\pm}^{(1)}$. On the contour $\Gamma^{(2)}$ we then get the modified problem $f_-^{-1}f_+ = \tilde{g}_-(\lambda)g^{(2)}(\lambda)g_-^1(\lambda)$. We can proceed in this manner until we have found the solution to our full Riemann-Hilbert problem. The remaining question is what $\tilde{g}^{(k)}(\lambda)$ should be in order for the solution to the Riemann-Hilbert problem to solve the equations (33) and (34). We already know that the flows associated to the M_i commute. Putting this together with the discussion above, we find:

$$\theta_-^{-1}(\lambda, t)\theta_+(\lambda, t) = e^{-g_+(\lambda, 0) \sum_i \xi_i(\lambda) t_i g_+^{-1}(\lambda, 0)} \quad (46)$$

as our Riemann-Hilbert problem. A similar calculation as done for the single pole case shows that this does indeed solve our hierarchy (33,34).

5 Conclusion

We have shown how the Riemann-Hilbert factorization problem is formulated as a mathematical theorem and seen how it arises in integrable systems. It is quite a surprising feature of integrable systems that hierarchies like (33,34) admit expression of their solution in terms of an algebraic problem. In this paper we discussed how this feature follows from the formulation of the hierarchy in terms of loop groups and algebras.

In general, there is no algorithm to solve a problem like (36). However, the hierarchy was a system of nonlinear differential equations and the factorization problem, as can be seen from the proof of (2.1), boils down to solving a linear singular-integral equation. So in a sense the formulation of the original problem in terms of a Riemann-Hilbert problem linearizes the problem. Many techniques have been developed to handle this problem, for an overview see [5].

In applications the Riemann-Hilbert problem is often solved by making a good Ansatz. A nice example of this is the open Toda chain treated in [2], where the problem is already formulated in terms of groups and algebras, making the procedure quite transparent.

References

- [1] D.V. Anosov and A.A. Bolibruch. *The Riemann-Hilbert Problem*. Aspects of Mathematics, E 22, 1994.
- [2] O. Babelon, D. Bernard, and M. Talon. *Introduction to Classical Integrable Systems*. Cambridge University Press, 2000.
- [3] Alain Connes and Dirk Kreimer. Renormalization in quantum field theory and the Riemann-Hilbert problem I: the Hopf algebra structure of graphs and the main theorem. *Commun. Math. Phys.* 210, 249-273, 2000.
- [4] John B. Conway. *A Course in Functional Analysis*. Springer, 2007.
- [5] Alexander R. Its. The Riemann-Hilbert problem and integrable systems. *Notices of the AMS*, Volume 50, Number 11, 2003.