1. TAC MT 23: The Six Functor Formalism

Organisers: Adri Marin and Thomas Wasserman

Summary. Our main source is Scholze's lecture notes [Sch22]. There are many other sources that give accounts of six-functor formalisms in specific contexts, see for example [Gal22] for a treatment with a view towards motivic homotopy theory. The main selling point of Scholze's description is that it *defines* what a six-functor formalism is abstractly, using a definition due to Mann [Man22].

The formalism presented is ∞-categorical in flavour. To aid accessibility, we will devote one lecture to covering the necessary background. We will also aim to use the example of the six-functor formalism for locally compact Hausdorff spaces as a handrail.

Schedule.

Lecture 1. **Introduction and overview.**
Aim: explain what a six-functor formalism roughly is, what goes into setting it up, and why one should care. In particular our guiding example of locally compact Hausdorff spaces will be sketched. This lecture will be based on the first (and some of the second) chapter of the lecture notes.
Speaker: Thomas.

Lecture 2. **∞-categorical preliminaries.**
Aim: introduce the necessary ∞-categorical background: symmetric monoidal ∞-categories, straightening/unstraightening, leading to symmetric monoidal ∞-categories of correspondences. Chapters II and III of the Scholze's notes summarise what is needed, and contain references to the relevant parts of Lurie's books [Lur06; Lur17].
Speaker: ?

Lecture 3. **Abstract six-functor formalism and construction.**
Aim: Discuss and unpack the abstract definition of a six-functor formalism, then discuss the construction of six-functor formalisms starting from some basic (geometrically motivated) assumptions. This starts by discussing the notion of a three-functor formalism (sprinkled through Chapters II and III), and then covers Chapter IV.
Speaker: Adri.

Lecture 4. **Poincaré duality.** Aim: state and prove the notion of Poincaré Duality for six-functor formalisms. This is Chapter V in the notes.
Speaker: ?

Lecture 5. **Results about six-functor formalisms.** Aim: discuss complementary results about abstract six-functor formalisms that make them easier to work with in practice: for which f does one have $f_* = f_!$ or $f^* = f^!$, when are objects isomorphic to their Verdier duals? This is Chapter VI from the notes.
Speaker: ?

Lecture 6. **Example: locally compact topological spaces.** Aim: setup the six-functor formalism for locally compact Hausdorff spaces, and discuss Poincaré Duality in this context. This is Chapter VII from the notes, with some ingredients from Chapter V.
Speaker: ?

Lecture 7. **Example: coherent sheaves.** Aim: discuss six-functor formalisms for coherent sheaves, and highlight the issues one encounters in doing this. This follows Chapter VIII from the notes.
Speaker: ?

Lecture 8. **Coherent sheaves revisited using condensed mathematics.** Aim: give a sketch of how using solid modules helps overcome the issues encountered when dealing with coherent sheaves. This requires summarising material from Scholze's notes on condensed mathematics [Sch19], an existing such summary is in [Leh22].
Speaker: Sofia?

References

[Gal22] Martin Gallauer. *An introduction to six-functor formalisms*. 2022. URL: https://homepages.warwick.ac.uk/staff/Martin.Gallauer/docs/m6ff.pdf.

[Leh22] Simon Pepin Lehalleur. "Six-functor formalism; why solid abelian groups?" In: (2022). URL: https://webspace.science.uu.nl/~dobbe012/doc/six-functors.pdf.

[Lur06] Jacob Lurie. "Higher Topos Theory". In: *Higher Topos Theory* (Aug. 2006), pp. 1–925. URL: https://arxiv.org/abs/math/0608040v4.

[Lur17] Jacob Lurie. *Higher Algebra*. 2017. URL: https://www.math.ias.edu/~lurie/papers/HA.pdf.

[Man22] Lucas Mann. "A p-Adic 6-Functor Formalism in Rigid-Analytic Geometry". In: *ArXiV:2206.02022* (June 2022). URL: https://arxiv.org/abs/2206.02022v1.

[Sch19] Peter Scholze. *Lectures on Condensed Mathematics*. 2019. URL: https://www.math.uni-bonn.de/people/scholze/Condensed.pdf.

[Sch22] Peter Scholze. *Six-Functor Formalisms*. 2022. URL: https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf.