1. Topology Advanced Class MT 24: Factorisation Algebras

Organiser: Thomas Wasserman[1]

Goal. The goal of this class is to get familiar with the Factorisation Algebra formalism. We will first look at factorisation algebras as a mathematical formulation of Quantum Field Theory (QFT), following Costello and Gwilliam [CG16; CG21]. We will then move on to studying factorisation algebras in the topological setting, this is also known as factorisation homology.

Overview. Factorisation algebras are an approach to mathematically formalising QFT on a fixed manifold. A factorisation algebra is a gadget that assigns to an open in the manifold a chain complex, thought of as encoding the observables of some QFT in that open. If one can make an observation in an open, one can certainly also make it in any open containing it – this translates to factorisation algebras having the structure of a pre-cosheaf. Additionally, simultaneous observation gives a way of combining observations on disjoint opens, this leads to a kind of algebraic structure. All this data satisfies a locality condition known as Weiss codescent, this ensures that a factorisation algebra can be glued together from what it does on neighbourhoods of finitely many points. We will examine this definition in detail and see some examples of how to construct factorisation algebras.

Factorisation algebras that assign the same value to every configuration of a number of disks in the manifold are called locally constant. Locally constant factorisation algebras correspond to topological quatum field theories, and have been studied in the topology literature under the name factorisation homology. We will see how Costello–Gwilliam factorisation algebra and topologists' factorisation homology compare, and spend a couple of lectures learning about factorisation homology in depth: there are neat features resembling the Mayer–Vietoris theorem and Poincaré duality of ordinary (co)homology.

Prior knowledge. The aim is to make this class accessible to graduate students in topology, but people with different backgrounds are warmly welcome. Do not be afraid to speak up if you think some material is insufficiently covered so either the speaker for the week or the organiser can make sure everyone is able to follow.

We will assume a working knowledge of manifolds, categories, basics of sheaf theory, homological algebra, and basics of topological vector spaces. For the factorisation homology part of the class we will often work with ∞-categories, and briefly introduce what is needed along the way. No prior knowledge of physics will be assumed for this class.

If you want to be prepared for the class, the appendices to [CG16] provide some of this background. If there are particular topics you would like help with, let Thomas, André or Lukas know.

Sources. Our main source for the first part of the class is Costello and Gwilliam's introductory book [CG16], with a selection of topics from [CG21]. These books are accessible online through logging in with your institutional account – let the organiser know if you have trouble finding them or getting access. For factorisation homology we will mainly rely on [AF20]. Other good sources are Ginot's lecture notes [Gin14] and Lurie's treatment of chiral homology in Higher Algebra [Lur17].

Lectures. Graduate students are strongly encouraged to give one of the lectures. The content of each talk is outlined below, but the speaker has creative licence in their approach, selection of examples or additional topics. Just coordinate with the organiser.

The speaker for the week is expected to prepare a lecture of one hour and a quarter. This leaves a quarter of an hour for questions and discussion that we will use to ensure that everyone is following along and getting the most out of the class. The audience will keep in mind that (especially more junior) speakers will be covering material they have only just learned, and we will collectively do our best to make this class a good learning experience for everyone involved. The organiser will provide private feedback on the talk afterwards, the advanced class also serves as an opportunity to develop presentation skills.

While preparing a lecture do not hesitate to contact the organiser (or one of the other more senior participants). The organiser is expected to keep a couple of hours a week aside to help with preparation.

Get in touch if you want to give a talk!

Broadening. The class can be used towards filling the broadening requirement for DPhil students by giving one of the lectures.

Time and place. We meet Monday mornings at 11am in one of the classrooms in the mezzanine of the maths department, check the screens.

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Schedule.

Week 1 **Introduction.** (*Speaker: Thomas*) Overview of Factorisation Algebras, motivation and background from a mathematical (and in particular topological) perspective, and background on classical and quantum field theory, quantisation, and observables and correlators.

Week 2 **Pre-factorisation algebras.** *(Speaker: Paul)* The first step towards defining factorisation algebras is defining pre-factorisation algebras. These are already interesting gadgets in their own right – being a factorisation algebra is a property of a pre-factorisation algebra. This lecture will introduce pre-factorisation algebras following Chapter 3 in [CG16], including the main example of the factorisation envelope of a sheaf of Lie algebra. The required analysis can be black-boxed for this lecture.

Week 3 **Free field theories.** (*Speaker: Nivedita*) In this lecture we will see pre-factorisation algebras in action by looking at the examples of free field theories, following Chapter 4 in [CG16]. This lecture will cover how to construct the pre-factorisation algebra associated to a free classical field theory, and how to quantise this to the pre-factorisation algebra for the associated free quantum field theory.

Week 4 **Factorisation algebras.** (*Speaker: Nathan*) This lecture will introduce the codescent condition that factorisation algebras satisfy, and verify this for the pre-factorisation algebras discussed in the preceeding lectures. This essentially covers Chapter 6 from [CG16], skipping sections 6.3 and 6.4.

Week 5 **Noether's Theorem.** (*Speaker: Glen*) This will be the final lecture on factorisation algebras and quantum field theory. To showcase the richness of the formalism, it will cover the advanced topic of Noether's Theorem for factorisation algebras. This lecture will give an overview of Part III of [CG21], necessarily with limited technical detail.

Week 6 **From locally constant factorisation algebras to E_n-algebras.** (*Speaker: Sam*) This lecture transitions from factorisation algebras to factorisation homology. This would be an expanded version of section 6.4 in [CG16], and would cover a sketch of a proof of the equivalence between E_n algebras and locally constant factorisation algebras on $\mathbb{R}^n$ (following [Lur17, Section 5.4]) as well as sections 6.4.2 and 6.4.3 from [CG16].

Week 7 **Factorisation homology and $\otimes$-excision** (*Speaker: Adri*) This lecture will define factorisation homology and explain the Mayer–Vietoris-like $\otimes$-excision that factorisation homology satisfies. This covers sections 2 and 3 of [AF20]. The purely topological counterpart to this is [AF15], this might prove more accessible.

Week 8 **Non-abelian Poincaré duality.** (*Speaker:*) The goal of this lecture is to discuss non-abelian Poincaré duality for factorisation homology, and give a sketch of a proof. This covers section 4 of [AF20] (with topological counterpart in [AF15]), but the speaker is invited to also consider discussing Theorem 7.8 from [AF20].

References

[AF15] David Ayala and John Francis. "Factorization homology of topological manifolds". In: *J. Topol.* 8.4 (2015), pp. 1045–1084. ISSN: 1753-8416. DOI: `10.1112/jtopol/jtv028`. URL: `https://doi.org/10.1112/jtopol/jtv028` (visited on 12/06/2023).

[AF20] David Ayala and John Francis. "Handbook of Homotopy Theory". In: Chapman and Hall/CRC, 2020. Chap. A factorization homology primer, pp. 39–101. DOI: `10.1201/9781351251624-2`.

[CG16] K. Costello and O. Gwilliam. *Factorization Algebras in Quantum Field Theory. Volume 1.* Vol. 1. New Math. Monogr. Cambridge: Cambridge University Press, 2016. ISBN: 9781107163102. DOI: `10.1017/9781316678626`. URL: `https://www.cambridge.org/core/books/factorization-algebras-in-quantum-field-theory/9597AFE8E8767F8F38A73C74B8F2501B`.

[CG21] Kevin Costello and Owen Gwilliam. *Factorization Algebras in Quantum Field Theory. Volume 2.* English. Vol. 41. New Math. Monogr. Cambridge University Press, 2021. ISBN: 978-1-107-16315-7; 978-1-00-900616-3; 978-1-316-67866-4. DOI: `10.1017/9781316678664`.

[Gin14] Grégory Ginot. "Notes on factorization algebras, factorization homology and applications". In: *arXiv:1307.5213* (2014). arXiv:1307.5213 [math] type: article. DOI: `10.48550/arXiv.1307.5213`. URL: `http://arxiv.org/abs/1307.5213` (visited on 09/30/2024).

[Lur17] Jacob Lurie. *Higher Algebra.* https://www. math. ias. edu/ lurie/papers/HA. pdf, 2017. URL: `https://www.math.ias.edu/~lurie/papers/HA.pdf`.