Research Statement - Thomas Wasserman

My research concerns Topological Quantum Field Theory (TQFT). This a young topic in Mathematics, that links Topology, Algebra and Physics. My thesis [Was17a] and papers [Was17b, Was17c, Was17d] study aspects of Topological Gauge Theory (TGT), and in collaborative work I am exploring applications of TQFT to Algebraic Geometry and Mathematical Physics, as well as developing some tools in Category Theory. In the coming years, I will continue this line of research.

Background: Topological Quantum Field Theory

In his landmark paper [Wit89], Witten showed how to obtain the Jones polynomial, a well-known knot invariant, from a quantum field theory, known as Chern-Simons theory. Realising the potential of this discovery, Atiyah gave a definition [Ati88] of what he called Topological Quantum Field Theory. His idea was to view theories such as the one that Witten had described as functors from a category of cobordisms to the category of vector spaces, as illustrated in Figure 1.

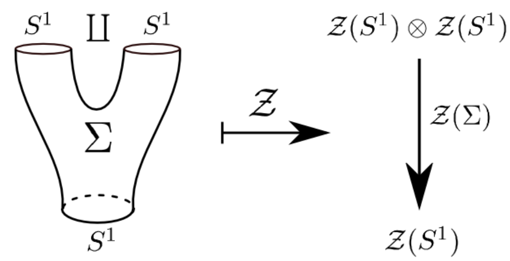

Figure 1: Atiyah's definition of a TQFT $\mathcal{Z}$.

Background: Once-Extended Three-Dimensional TQFTs

To fit Chern-Simons theory into Atiyah's approach requires one to consider once-extended TQFTs in three dimensions. This is a categorification of Atiyah's definition where one also allows cobordisms between cobordisms as an extra layer of morphisms, called 2-morphisms. In particular, for a three-dimensional theory, objects are then circles, morphisms are surfaces and 2-morphisms are three-dimensional manifolds. The target is similarly categorified, one considers the category of categories enriched in (and tensored over) vector spaces.

Recently, a complete classification [BDSPV15] was given of such TQFTs as being in one-to-one correspondence with modular tensor categories (MTCs). This correspondence takes a TQFT to its value on the circle. A fascinating aspect of this result is that it gives a bridge between topology and algebra, that allows us to view the algebraic structure of these MTCs through the study of cobordisms.

MTCs are particular examples of monoidal categories $(\mathcal{C}, \otimes)$ equipped with a braiding: natural isomorphims $a \otimes b \xrightarrow{\beta_{a,b}} b \otimes a$. This braiding is required to be maximally non-symmetric: only the monoidal unit and its direct sums are transparent in the sense that the double braiding $(\beta_{b,a} \circ \beta_{a,b})$ with it and any other object is trivial. Without this requirement, we call such a category premodular. The failure of a premodular category to be modular is measured by its subcategory of transparent objects.

The category of TQFTs on a given bordism category with a given target carries a symmetric monoidal structure. This tensor product of TQFTs is given by taking the monoidal product of the values of the TQFTs. In the oriented once-extended three-dimensional case, this product is, at the level of the MTCs, given by the so-called Deligne tensor product. The Deligne tensor product of two linear categories $\mathcal{C}$ and $\mathcal{D}$ has objects $c \boxtimes d$, with $c \in \mathcal{C}$ and $d \in \mathcal{D}$, and as hom-spaces the tensor product (in vector spaces) of the hom-spaces in $\mathcal{C}$ and $\mathcal{D}$.

Goal: Classify G-Equivariant Topological Quantum Field Theories

As a variation on oriented once-extended three-dimensional TQFT, one can equip the manifolds in the bordism category with principal G-bundles for some finite group G. TQFTs on this bordism category are called G-equivariant Topological Quantum Field Theories. In physics these are used to describe condensed matter systems with a so-called topological order with finite local symmetry [LKW16]. Using a procedure called orbifolding (discussed below) such a G-equivariant theory gives rise to an oriented TQFT. An oriented TQFT obtained in this way is called a Topological Gauge Theory, and G is then called the gauge group.

The prototypical example of a TGT is Dijkgraaf-Witten theory [DW90]. In its simplest form, this is the oriented once-extended three-dimensional TQFT that assigns to the circle the so-called Drinfeld centre $\mathcal{Z}(\mathrm{Rep}(G))$ of the representation category of the finite group.

One of my goals for the coming years to ***give a classification of once-extended three-dimensional G-equivariant TQFTs.*** At present, I am working on ***giving a generator and relation presentation of the once-extended three-dimensional bordism category equipped with principal G-bundles***, building on work from [BDSPV14]. In his book [Tur10], Turaev constructs a large class of TGTs from so-called G-crossed modular categories. The expected classification result is that these encompass all possible TGTs.

The classification theorem from [BDSPV15] is an instantiation of a more general result detailed in [BDSPV14]. Roughly, this general theorem states that in any symmetric monoidal bicategory one has a notion of modular object, a generalisation of MTC. A particular bicategory that is interesting from the point of view of TGT is the 2-category of categories enriched in a symmetric fusion category $\mathcal{A}$. In future work, I will ***classify the modular objects in 2-categories of categories enriched over a symmetric fusion category.*** The expected result is that these modular objects are $\mathcal{Z}(\mathcal{A})$-crossed braided categories, as defined in [Was17a], satisfying a non-degeneracy condition.

Goal: Understand Orbifolding

Given a G-equivariant TQFT, one can produce an oriented TQFT by a procedure called orbifolding. This procedure comes from physics, where the oriented theory obtained in this way is the one of interest. This procedure is not well-understood in general. Some inroads have been made for the case of non-extended TQFTs, see [SW17]). Despite this lack of precise understanding, in my thesis work I was able to make parts of this construction and its consequences precise. In future work, I will use results from my thesis to ***understand orbifolding in the case of once-extended three-dimensional topological quantum field theories.***

G-equivariant TQFTs can be build from the data of a so-called G-crossed modular braided category. These are G-graded monoidal category categories equipped with a G-action and a twisted braiding. The tensor product of such TQFTs corresponds to the degree-wise Deligne tensor product $\underset{G}{\boxtimes}$. Given a G-equivariant theory, orbifolding should produce an oriented theory. On the G-crossed modular categories, this corresponds to taking so-called homotopy fixed points, and is usually called equivariantisation. In [DGNO10], this procedure is shown to define a 2-equivalence $\mathbf{Eq}$, with inverse $\mathbf{DeEq}$ between a 2-category G-$\mathbf{XBF}$ of G-crossed braided categories and a 2-category $\mathbf{BFC}/\mathcal{A}$ of braided fusion categories containing the symmetric fusion category $\mathcal{A} = \mathrm{Rep}(G)$.

When considering TGTs, the Deligne tensor product of the MTCs that define them is not the appropriate tensor product, as the resulting TQFT would be a TGT with $G \times G$ as gauge group. Rather, we should find the image of the degree-wise tensor product under the orbifold construction. This image will be the reduced tensor product.

Result: Reduced Tensor Product

The main goal of my thesis [Was17a] was to define a symmetric monoidal product $\underset{\text{red}}{\boxtimes}$ which corresponds to the appropriate way of tensoring TGTs. This product $\underset{\text{red}}{\boxtimes}$, called the reduced tensor product, on $\mathbf{BFC}/\mathcal{A}$ makes the 2-functors $\mathbf{DeEq}$ and $\mathbf{Eq}$ into symmetric monoidal 2-functors with respect to $\underset{G}{\boxtimes}$. The reduced product is directly computable in terms of the monoidal structure and the braiding of these braided fusion categories. Additionally, my construction avoids assuming that $\mathcal{A}$ is a representation category, treating the so-called Tannakian and super-Tannakian (see [Del90, Del02]) cases on equal footing. In order to define this tensor product, I introduce the novel notion of a $\mathcal{Z}(\mathcal{A})$-crossed braided category, where the braided fusion category $(\mathcal{Z}(\mathcal{A}), \otimes_c)$ is the so-called Drinfeld centre of $\mathcal{A}$. The 2-category $\mathcal{Z}(\mathcal{A})$-$\mathbf{XBF}$ of such categories comes equipped with a natural symmetric monoidal structure $\underset{s}{\boxtimes}$, and fits into the following triangle (or a super-version):

$$
\begin{tikzcd}
& (\mathcal{Z}(\mathcal{A})\text{-}\mathbf{XBF}, \underset{s}{\boxtimes}) \arrow[dl, shift left, "\mathbf{DeEnrich}"] \arrow[dr, shift left, "\overline{\overline{(-)}}"] \arrow[d, phantom, "(A)"] & \\
(\mathbf{BFC}/\mathcal{A}, \underset{\text{red}}{\boxtimes}) \arrow[ur, shift left, "\underline{(-)}"] \arrow[rr, shift left, "\mathbf{DeEq}"] & {} & (G\text{-}\mathbf{XBF}, \underset{G}{\boxtimes}), \arrow[ll, shift left, "\mathbf{Eq}"] \arrow[ul, shift left, "\mathbf{Fix}"]
\end{tikzcd}
$$

where $\mathcal{C} \underset{\text{red}}{\boxtimes} \mathcal{D} := \mathbf{DeEnrich}(\underleftarrow{\mathcal{C}} \underset{s}{\boxtimes} \underleftarrow{\mathcal{D}})$ is the reduced tensor product. This product owes its name to Drinfeld, who defined it in the case where $\mathcal{A} = \mathbf{sVect}$ in [Drib].

In my thesis I constructed the top part of this triangle. The main results of my thesis can be summarised as:

Theorem A ([Was17a, Was17b, Was17c, Was17d]). *The triangle (A) above is a commutative triangle with mutually inverse symmetric monoidal 2-equivalences as edges.*

The main ingredient for this Theorem is the observation that the Drinfeld centre $\mathcal{Z}(\mathcal{A})$ of the symmetric fusion category $\mathcal{A}$ admits a second monoidal structure $\otimes_s$ that is symmetric, see [Was17d]. In [Was17c], I show that this monoidal structure is bilaxly compatible with the usual braided monoidal structure $\otimes_c$, making $\mathcal{Z}(\mathcal{A})$ into a bilax 2-fold monoidal category. A $\mathcal{Z}(\mathcal{A})$-crossed braided category is then defined in my thesis to be a category enriched over $(\mathcal{Z}(\mathcal{A}), \otimes_s)$, with a braided monoidal structure that factors through the product of $\mathcal{Z}(\mathcal{A})$-enriched categories $\underset{c}{\boxtimes}$ that uses $\otimes_c$ on the hom-objects. The product $\underset{s}{\boxtimes}$ is similarly uses $\otimes_s$ on hom-objects. The 2-functor $\underleftarrow{(-)}$ takes a braided fusion category $\mathcal{C}$ containing $\mathcal{A}$ and enriches it over $\mathcal{Z}(\mathcal{A})$. This enrichment is used to encode the braiding behaviour of the objects of $\mathcal{C}$ with respect to the objects of $\mathcal{A}$, and the swap functor for $\underset{c}{\boxtimes}$ takes this into account. This ensures that the resulting object is genuinely braided, rather than braided up to some twist. The construction $\underleftarrow{(-)}$ is done purely in terms of the monoidal structure and braiding for $\mathcal{C}$.

As this work describes the appropriate tensor product of TGTs, it also aids our understanding of TGTs.

Goal: Explicit Formulae for the Structure of Minimal Modular Extensions

Let $\mathcal{C}$ be a premodular category, its failure to be modular is measured by its category of transparent subobjects $\mathcal{A}$. A modular category $\mathcal{M}$ containing $\mathcal{C}$ is called a modular extension of $\mathcal{C}$, and minimal if the subcategory of objects of $\mathcal{M}$ transparent with respect to $\mathcal{A}$ is precisely $\mathcal{C}$. We denote the set of such minimal modular extensions by $\mathrm{MME}(\mathcal{C}, \mathcal{A})$. This set is possibly empty, as was shown by Drinfeld [Dria]. If non-empty, $\mathrm{MME}(\mathcal{C}, \mathcal{A})$ is a torsor for $\mathrm{MME}(\mathcal{A}, \mathcal{A})$ as was shown abstractly in [LKW16]. Shortly thereafter, a partial procedure in the case $\mathcal{A} = \mathbf{sVect}$ was given in [BGH$^+$16] to compute this action of $\mathrm{MME}(\mathcal{A}, \mathcal{A})$ explicitly.

The action $\mathrm{MME}(\mathcal{A}, \mathcal{A}) \times \mathrm{MME}(\mathcal{C}, \mathcal{A}) \to \mathrm{MME}(\mathcal{C}, \mathcal{A})$ corresponds to taking the reduced tensor product, defined in my thesis, of the elements in the pair. This will allow me to, in future work, ***read off explicit formulas for the structure data of the resulting minimal modular extension,*** recovering in particular the results from [BGH$^+$16].

Furthermore, the reduced tensor product extends to a product between minimal modular extension of different categories. This product structure on minimal modular extensions has been overlooked in the literature, and is interesting to study in its own right. In particular, in collaborative work with the physicist Slingerland, I will use this product to generate new modular tensor categories.

As mentioned above, the set of minimal modular extensions $\mathrm{MME}(\mathcal{C}, \mathcal{A})$ of a given category may be empty. It is an open question under what conditions on $\mathcal{C}$ this set is non-empty. From the point of view of topological gauge theory, assuming $\mathcal{A} \cong \mathrm{Rep}(G)$, this is the question whether there is a TGT that comes from orbifolding a G-equivariant TQFT such that the so-called modularisation of $\mathcal{C}$ is the MTC that the field theory assigns to the circle equipped with trivial G-bundle. From this point of view, we are solving an extension problem. One can try to solve this using symmetric monoidal Kan extensions, which I am developing in joint work with Penney. With this, I can ***determine conditions for when a braided fusion category admits a modular extension.***

Collaborative work

- **The Landau-Ginzburg Conformal Field Theory Correspondence and Symmetric Monoidal Bicategories.** *Joint work with Ana Ros Camacho, currently Marie Sklodowska-Curie Individual Fellow at Utrecht University.* Ros Camacho's research is on a correspondence, first observed by physicists, between conformal field theories and quantum field theories. The most pressing open question in this field is to give a mathematically precise conjecture describing this correspondence, which can then be proved. ***The goal of our collaboration is to give a higher category framework in which to state a conjecture capturing the Landau-Ginzburg Conformal Field Theory correspondence.***

- **1+1 TQFT defined from the mixed Hodge structure character varieties for Riemann surfaces** *Joint work with Ángel González Prieto, currently PhD. student at Complutense University, Madrid.* In the context of Mirror Symmetry in Geometry, one is interested in studying the character varieties of monodromy representations of fundamental groups of surfaces. In particular, one would like to compute invariants such as the so-called e-polynomial of these spaces. This is in general a hard problem, but González Prieto has been able to reduce the computations involved to computing e-polynomials on elementary building blocks of the surfaces. This gives a symmetric lax monoidal functor out of the two-dimensional oriented bordism category, rather than an honest TQFT. Motivated by this will we ***show that, given a functor defined on the subcategory of the two dimensional bordism category generated by discs, cylinders and the torus with two discs removed, one can always obtain one of these "lax TQFTs".***

- **Finding fully extended two-dimensional G-equivariant TQFTs.** *Joint work with Kursat Sozer, currently PhD. student at Indiana University.* Sozer is working on extending the classification of oriented two-dimensional fully extended TQFTs (see [SP11]) in terms of seperable symmetric Frobenius algebras to a classification of G-equivariant theories, for a finite group G. The expected result is that these are classified by so-called strictly graded biangular G-algebras. These algebraic objects, however, are hard to find. We propose to give a construction similar to the so-called de-equivariantisation studied in [Was17a], to ***produce these G-algebras from seperable symmetric Frobenius algebras containing the group algebra $\mathbb{C}[G]$.***

- **The reduced tensor product, modular data and applications to Physics** *Joint work with Joost Slingerland, currently Lecturer at the National University of Ireland, Maynooth.* The reduced tensor product defined in [Was17a] is computable in terms of the modular structure and the braidings of the braided fusion categories. For MTCs, the Verlinde formula (see for example [RSW09]) determines the modular structure in terms of the braidings and the so-called twists. The data of the modular stucture and the twists is called the modular data. Part of Slingerland's work is on using algorithms to find new MTCs. We will ***determine a formula for the reduced tensor product in terms of the modular data, and use this to generate new modular tensor categories.***

- **Monoidal Kan extensions.** *Joint work with Mark Penney, currently Post Doc at the Max Planck Institute for Mathematics, Bonn.* TQFTs are by definition symmetric monoidal functors on bordism categories. Suppose that one is in the situation where one knows how to define such a TQFT on only a part of the bordism category, and one wishes to extend this to a TQFT on the whole bordism category. This kind of extension problem has a universal solution known as the Kan extension. In the context of ordinary categories and their functors this is a well-known tool, but this does not suffice in the case of TQFTs, where one needs to take into account the monoidality of the functors involved. Penney and I propose to carry out a recipe from [MT08], outlining how to compute Kan extensions of monoidal functors. This will require us find a good description of the bi-category of profunctors of monoidal categories. ***The goal is to find conditions for monoidal Kan extensions to exist, and, give explicit formulas to compute them.***

- **Homology Stability and TQFTs.** *Joint work with Renee Hoekzema, currently DPhil. student at the University of Oxford.* Given a sequence of groups $\{G_i\}$ with inclusions between them, we get a sequence of bordism categories of manifolds equipped with principal G_i-bundles, and functors between these. In applications to physics, see for example [DVV97] for the case with permutation groups, one is often interested in the limiting behaviour for a family of TGTs obtained from equivariant TQFTs on these bordism categories. (Invariants of) these TGTs can be linked to the (co)homology of the G_i. For example, the invertible equivariant theories are classified by low degree cohomology groups. ***The goal is to find (invariants of) sequences of TGTs that stabilise.***

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