Gravitational Waves from Bubble Collisions during First-Order Phase Transitions

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Abstract

Written for advanced bachelor students in physics with a minimal knowledge of General Relativity and cosmology, this bachelor thesis gives an introduction into the subject of gravitational waves. It treats the waves on both flat and cosmological background and derives a formula for the energy-density spectrum due to gravitational waves. We then apply the theory developed to gravitational waves from bubble collisions during first-order phase transitions that occur in the radiation era, and find that the energy-density scales as the square of the ratio between the duration of the phase transition and the Hubble time at the time of the phase transition and as the square of the ratio between the kinectic energy-density associated with the phase transition and total energy-density at that time.

Contents

1 Introduction

This text is written as a bachelor thesis in physics, as part of the bachelor seminar theoretical physics in the academic year '07-'08. The goal of this seminar was to let a group of students together discover the main principles of cosmology. This was done by letting each student pick a subject to prepare a presentation on, and to write a thesis about. Being already familiar with General Relativity, I picked the subject of gravitational waves, along with two other students. We divided the subject: a treatment of gravitational waves from astrophysical sources and methods of detection is found in [7], and a treatment of the evolution of long wavelength gravitational waves created during inflation is given in [10]. In this thesis I focus on shorter wavelength gravitational waves created by bubble collisions during first-order phase transitions, and I aim to give the reader insight into how the relevant physical parameters give rise to gravitational waves.

These gravitational waves are of physical importance for they could provide us with a view of the phase transitions in the early universe (for a treatment of these phase transitions see [13]), in particular the electroweak phase transition. These phase transitions are hidden from our sight, because the universe was opaque at the time they took place. Gravitational waves, however, have the interesting property that they barely interact with anything, thus allowing them to pass barely attenuated from the phase transition to us. This property of course also means that they are very hard to detect, but promising efforts are being made to build detectors that are sensitive enough.

As this text is written as a bachelor thesis, I have tried to make it suitable for physics students at the end of their bachelor, with a basic knowledge of cosmology, as taught in the introductory lectures of the bachelor seminar. Calculations are therefore presented at a high level of detail, allowing the reader to focus on the physics instead of the computations. However, since gravitational waves are a prediction from General Relativity, I have chosen to assume the reader has at least a basic knowledge of this theory, even though it is not part of the bachelor curriculum. Starting from this assumption, I first present the linearized theory of relativity, which predicts the existence of gravitational waves, in the relatively simple setting of a flat background on which the waves propagate, with as main goal to give the reader some insight and intuition for what gravitational waves are, and where they come from. The next chapter gives a quick recapitulation of the basics of Friedmann-Lemaître-Robertson-Walker cosmology as presented during the introductory lectures of the seminar. The reader should then be ready for the somewhat more complicated derivation of the equations of motion for gravitational waves in a FLRW-universe. In chapter 5 we then present a scheme to extract the main observable quantity of gravitational waves, namely the spectrum of abundance of energy density, from the theory. Finally, in the last chapter, we get to our intended subject, gravitational waves from bubble collisions during first-order phase transitions, where I hope to give the reader insight into the process underlying the generation of gravitational waves by bubble collisions.

Throughout the text we use the following conventions:

- index convention: Latin indices run from 1 to 3, denoting the spatial coordinates, and Greek indices run from 0 to 3, with 0 denoting the time coordinate,

- sign convention:
$$\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1), \tag{1}$$

- Fourier convention:
$$\hat{f}(k) = \int \frac{dx}{\sqrt{2\pi}} f(x) e^{ikx}, \tag{2}$$

- form for the Einstein equations (to fix the signs):
$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{3}$$

- Einstein summation convention (unless stated otherwise):
$$T_\mu B^\mu = \sum_{\mu=0}^{3} T_\mu B^\mu, \tag{4}$$

- and natural units:
$$\hbar = c = 1. \tag{5}$$

As the reader may sometimes need some distraction from the black and white text, I refer for decoration to figure 1.

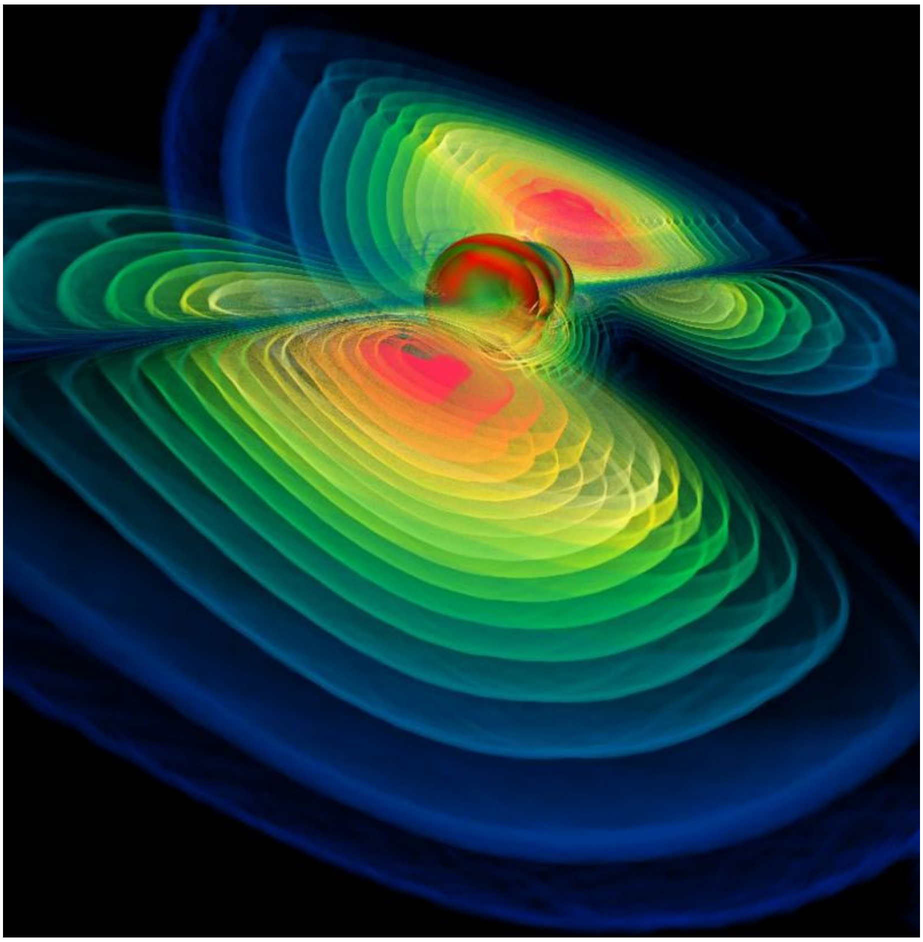

Figure 1: Decoration: colorful impression of gravitational waves.

2 Gravitational Waves

The possibility of existence of gravitational waves follows from the linearized version of General Relativity. In this approximation for the much more complicated equations of the full theory of General Relativity, one looks at small pertubations from known solutions to Einstein's equations. In this thesis we will consider pertubations on the Minkowski spacetime and on Friedmann-Lemaître-Robertson-Walker background.
To get some feel for what gravitational waves are, and how they follow from Einstein's theory of General Relativity, we will first go through the theory of linearized General Relativity on Minkowski space. This is the simpler case, and thus gives more insight at the cost of less work.

2.1 Linearized Einstein Equations

For the derivation of the linearized Einstein equations, we will be closely following [5]. The physical situation described by linearized General Relativity is that of a weak gravitational field.
The first step is to assume we have Minkowski space, with a small, symmetric tensor field perturbation on it:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{6}$$

where we assume $|h_{\mu\nu}| \ll 1$. Now we go through the usual procedure to find Einstein's equations, but we ignore anything higher than first order in h. We thus have for the inverse metric:

$$\delta^{\lambda}_{\mu} = g_{\mu\nu} g^{\nu\lambda} = \eta_{\mu\nu} g^{\nu\lambda} + h_{\mu\nu} g^{\nu\lambda}, \tag{7}$$

which is solved by

$$g^{\nu\lambda} = \eta^{\nu\lambda} - h^{\nu\lambda}. \tag{8}$$

Now that we have the inverse metric, we can calculate the Christoffel symbols:

$$\begin{aligned} \Gamma^{\rho}_{\mu\nu} &= \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) && (9) \\ &= \frac{1}{2} \eta^{\rho\sigma} (\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}), && (10) \end{aligned}$$

where we used $\partial_\sigma \eta_{\mu\nu} = 0$, and neglected all terms higher than first order in $h_{\mu\nu}$. Now on to the Riemann curvature tensor, for which we can shorten the calculation a bit by noting that the Γ^2-terms are always of quadratic order in $h_{\mu\nu}$, and can thus be neglected.

$$\begin{aligned} R^{\rho}{}_{\sigma\mu\nu} &= \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\sigma\mu} \\ &= \frac{\eta^{\rho\lambda}}{2} (\partial_\mu \partial_\sigma h_{\lambda\nu} + \partial_\mu \partial_\nu h_{\sigma\lambda} - \partial_\mu \partial_\lambda h_{\sigma\nu} \\ &- \partial_\sigma \partial_\mu h_{\lambda\nu} - \partial_\sigma \partial_\nu h_{\mu\lambda} + \partial_\sigma \partial_\lambda h_{\mu\nu}) \\ &= \frac{1}{2} \eta^{\rho\lambda} (\partial_\mu \partial_\nu h_{\sigma\lambda} - \partial_\mu \partial_\lambda h_{\sigma\nu} - \partial_\sigma \partial_\nu h_{\mu\lambda} + \partial_\sigma \partial_\lambda h_{\mu\nu}). && (11) \end{aligned}$$

Contracting over ρ and μ to obtain the Ricci tensor:

$$\begin{aligned} R_{\mu\nu} = R^{\kappa}_{\ \mu\kappa\nu} &= \frac{1}{2}\eta^{\kappa\lambda}(\partial_\kappa\partial_\nu h_{\mu\lambda} - \partial_\kappa\partial_\lambda h_{\mu\nu} - \partial_\mu\partial_\nu h_{\kappa\lambda} + \partial_\mu\partial_\lambda h_{\kappa\nu}) \\ &= \frac{1}{2}(\partial_\kappa\partial_\nu h_\mu^{\ \kappa} - \bar{\Box} h_{\mu\nu} - \partial_\mu\partial_\nu h + \partial_\mu\partial_\lambda h^\lambda_{\ \nu}), \end{aligned} \tag{12}$$

where we used $\bar{\Box}$ to denote the flat space D'Alembertian ($\bar{\Box} = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2 = \partial^\mu\partial_\mu$) and h to denote the trace of $h_{\mu\nu}$ ($h = h^\mu_{\ \mu} = \eta^{\mu\kappa}h_{\mu\kappa}$). The last ingredient we need for the Einstein equations is the Ricci scalar:

$$\begin{aligned} R = \eta^{\mu\nu}R_{\mu\nu} &= \frac{\eta^{\mu\nu}}{2}(\partial_\kappa\partial_\nu h_\mu^{\ \kappa} - \bar{\Box} h_{\mu\nu} - \partial_\mu\partial_\nu h + \partial_\mu\partial_\lambda h^{\lambda\mu}) \\ &= \frac{1}{2}(\partial_\kappa\partial_\nu h^{\nu\kappa} - \bar{\Box} h - \bar{\Box} h + \partial_\nu\partial_\lambda h^{\lambda\nu}) \\ &= \partial_\kappa\partial_\nu h^{\nu\kappa} - \bar{\Box} h. \end{aligned} \tag{13}$$

Now that we have all terms, we can write down the Einstein equations:

$$\begin{aligned} 8\pi G T_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} \\ &= \frac{1}{2}(\partial_\kappa\partial_\mu h^\kappa_{\ \nu} - \bar{\Box} h_{\mu\nu} - \partial_\nu\partial_\mu h + \partial_\nu\partial_\lambda h^\lambda_{\ \mu} \\ &- \eta_{\mu\nu}\partial_\kappa\partial_\lambda h^{\lambda\kappa} + \eta_{\mu\nu}\bar{\Box} h). \end{aligned} \tag{14}$$

Since the right hand side in these equations are first order in $h_{\mu\nu}$, so is $T_{\mu\nu}$, and we will thus only consider the lowest nonvanishing order in $T_{\mu\nu}$, which corresponds to zeroth order in $h_{\mu\nu}$.
As we will see, not all 10 degrees of freedom we have in $h_{\mu\nu}$ are physical, there is still gauge freedom left, and this needs dealing with before we start solving (14).

2.2 Gauge Invariance

To show we indeed have gauge freedom, we will consider the slightly more general case of a background space-time with perturbations on it, which is diffeomorphic to the physical space-time (still following [5]). In mathematical terms: let B be a pseudo-Riemannian manifold with metric $\eta_{\mu\nu}$ (for Minkowski space as background, but this discussion holds other metrics also), we will refer to this as the background space-time. Let P be a pseudo-Riemannian manifold equipped with some metric $g_{\mu\nu}$ that satisfies the Einstein equations, call this the physical space-time. Let ϕ: B $\rightarrow$ P be a diffeomorphism between the two. We can then define our perturbations to be tensor fields on B that are the difference between $\eta_{\mu\nu}$ and the pull-back of $g_{\mu\nu}$ (which can be seen as a representation of $g_{\mu\nu}$ on B):

$$h_{\mu\nu} = (\phi_* g)_{\mu\nu} - \eta_{\mu\nu}. \tag{15}$$

Now, before we had $|h_{\mu\nu}| \ll 1$, but for a general diffeomorphism, this does not hold for the $h_{\mu\nu}$ from (15). Of course, if the gravitational fields on P are weak, we can just restrict ourselves to diffeomorphisms for which this does hold. Then, by virtue of $g_{\mu\nu}$ satisfying the Einstein equations on P, $h_{\mu\nu}$ satisfies the linearized Einstein equations (14) on B.

We can use these facts to find out what gauge transformations leave the physical spacetime invariant. Take $X^\mu(x)$ to be a vector field on B. By means of its flow, any vector field generates a one-parameter family of diffeomorphisms H_ϵ: B $\rightarrow$ B, with $\epsilon \in$I, the maximal interval of definition. Note that for any diffeomorphism giving a small perturbation, $(\phi \circ H_\epsilon)$ will also give a small perturbation, if we take ϵ to be sufficiently small. We can thus use this flow to construct a family of perturbations, parameterized by ϵ. Simplifying as far as we can:

$$\begin{aligned} h^{(\epsilon)}_{\mu\nu} &= [(\phi \circ H_\epsilon)_* g]_{\mu\nu} - \eta_{\mu\nu} \\ &= [H_{\epsilon *}(\phi_* g)]_{\mu\nu} - \eta_{\mu\nu} \\ &= H_{\epsilon *}(h+\eta)_{\mu\nu} - \eta_{\mu\nu} \\ &= H_{\epsilon *}(h_{\mu\nu}) + H_{\epsilon *}(\eta_{\mu\nu}) - \eta_{\mu\nu}. \end{aligned} \tag{16}$$

Here we used $(f \circ k)_* = k_* \circ f_*$ for two diffeomorphisms f and k, equation (15), and the linearity of the pull-back. If we now approximate this under the assumption that ϵ is very small we see that the first order approximation with respect to ϵ for the last two terms is a Lie derivative (given by $\mathcal{L}_X f = \frac{d}{dt}|_{t=0} H_{t*} \circ f$) and $H_{\epsilon *}(h_{\mu\nu}) = h_{\mu\nu}$ at lowest order:

$$\begin{aligned} h^{(\epsilon)}_{\mu\nu} &= H_{\epsilon *}(h_{\mu\nu}) + \epsilon \left(\frac{H_{\epsilon *}(\eta_{\mu\nu}) - \eta_{\mu\nu}}{\epsilon} \right) \\ &= h_{\mu\nu} + \epsilon \mathcal{L}_X \eta_{\mu\nu}. \end{aligned} \tag{17}$$

We thus see we need the Lie derivative of the metric. To compute this, we need the expression of the Lie derivative in coordinates for a symmetric two-tensor:

$$\mathcal{L}_X g_{\mu\nu} = X^\sigma \partial_\sigma g_{\mu\nu} + (\partial_\mu X^\lambda) g_{\lambda\nu} + (\partial_\nu X^\lambda) g_{\lambda\mu}. \tag{18}$$

Plugging in the relations:

$$\partial_\sigma g_{\mu\nu} = \nabla_\sigma g_{\mu\nu} + \Gamma^\lambda_{\sigma\mu} g_{\lambda\nu} + \Gamma^\lambda_{\sigma\nu} g_{\lambda\mu} \tag{19}$$

and

$$\partial_\mu X^\lambda = \nabla_\mu X^\lambda - \Gamma^\lambda_{\mu\rho} X^\rho, \tag{20}$$

we get:

$$\begin{aligned} \mathcal{L}_X g_{\mu\nu} &= X^\sigma \nabla_\sigma g_{\mu\nu} + X^\sigma \Gamma^\lambda_{\sigma\mu} g_{\lambda\nu} + X^\sigma \Gamma^\lambda_{\sigma\nu} g_{\lambda\mu} + (\nabla_\mu X^\lambda) g_{\lambda\nu} \\ &- \Gamma^\lambda_{\mu\rho} X^\rho g_{\lambda\nu} + (\nabla_\nu X^\lambda) g_{\lambda\mu} - \Gamma^\lambda_{\nu\rho} X^\rho g_{\lambda\mu} \\ &= X^\sigma \nabla_\sigma g_{\mu\nu} + (\nabla_\mu X^\lambda) g_{\lambda\nu} + (\nabla_\nu X^\lambda) g_{\lambda\mu}. \end{aligned} \tag{21}$$

By the metric compatibility of the Levi-Cevita connection we get from this, in the case that $g_{\mu\nu}$ is our metric:

$$\mathcal{L}_X g_{\mu\nu} = \nabla_\mu V_\nu + \nabla_\nu V_\mu = 2\nabla_{(\mu} V_{\nu)}. \tag{22}$$

Now evaluating (17) to linear order, also ignoring mixed $h_{\mu\nu}$ and ϵ terms leads us to:

$$h^{(\epsilon)}_{\mu\nu} = h_{\mu\nu} + 2\epsilon \partial_{(\mu} X_{\nu)}, \tag{23}$$

because the covariant derivative is just the partial derivative in this setting. We have thus found a transformation that leaves the physics invariant, namely adding a $2\epsilon\partial_{(\mu}X_{\nu)}$ term to the pertubation, with ϵ small. We could verify that this transformation indeed leaves the physics invariant by checking that it does not change the linearized Riemann tensor, but this a straigthforward calculation which is not strictly necessary. What is important is that we have some gauge degree of freedom, and in the next section we will look at a clever way of using this freedom.

2.3 Fixing a Gauge

It turns out that there is a gauge which simplifies (14) somewhat [5]. This gauge is sometimes called the harmonic gauge, like in [5], and sometimes the linearized De Donder gauge, as in [11], and it imposes the condition $\Box x^\mu = \nabla^\lambda\nabla_\lambda x^\mu = 0$ on the coordinate functions. One can express this condition in terms of the metric and its derivatives (note that the x^μ are just functions, not components of a vectorfield):

$$\begin{aligned} 0 &= \nabla^\lambda\nabla_\lambda x^\mu \\ &= g^{\kappa\lambda}\nabla_\kappa\partial_\lambda x^\mu \\ &= g^{\kappa\lambda}\partial_\kappa\delta^\mu_\lambda - g^{\kappa\lambda}\Gamma^\alpha_{\kappa\lambda}\delta^\mu_\alpha \\ &= g^{\kappa\lambda}\Gamma^\mu_{\kappa\lambda}. \end{aligned} \tag{24}$$

So, in the case of linearized General Relativity on a Minkowski background:

$$\begin{aligned} 0 &= \frac{1}{2}\eta^{\kappa\lambda}\eta^{\mu\alpha}(\partial_\kappa h_{\lambda\alpha} + \partial_\lambda h_{\kappa\alpha} - \partial_\alpha h_{\kappa\lambda}) \\ &= \partial_\kappa h^{\kappa\mu} - \frac{\eta^{\mu\alpha}}{2}\partial_\alpha h. \end{aligned} \tag{25}$$

Contracting with $\eta_{\mu\nu}$ to get rid of the $\eta^{\mu\alpha}$:

$$\partial_\kappa h^\kappa_{\ \nu} - \frac{1}{2}\partial_\nu h = 0. \tag{26}$$

Note that these are just four conditions, so we still have six degrees of freedom left. Now, before we deal with them, we will have a look at what form the Einstein equations (14) get in this gauge:

$$\begin{aligned} 8\pi G T_{\mu\nu} &= \frac{1}{2}(\partial_\kappa\partial_\mu h^\kappa_{\ \nu} - \bar{\Box} h_{\mu\nu} - \partial_\nu\partial_\mu h + \partial_\nu\partial_\lambda h^\lambda_{\ \mu} \\ &\quad -\eta_{\mu\nu}\partial_\kappa\partial_\lambda h^{\lambda\kappa} + \eta_{\mu\nu}\bar{\Box} h) && (27) \\ &= \frac{1}{2}(\frac{1}{2}\partial_\mu\partial_\nu h + \frac{1}{2}\partial_\nu\partial_\mu h - \partial_\nu\partial_\mu h \\ &\quad -\bar{\Box} h_{\mu\nu} - \eta_{\mu\nu}\eta^{\kappa\alpha}\partial_\kappa\partial_\lambda h^\lambda_{\ \alpha}) && (28) \\ &= \frac{1}{2}(\eta_{\mu\nu}\bar{\Box} h - \bar{\Box} h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\kappa\alpha}\partial_\kappa\partial_\alpha h) && (29) \\ &= \frac{1}{4}\eta_{\mu\nu}\bar{\Box} h - \frac{1}{2}\bar{\Box} h_{\mu\nu}, && (30) \end{aligned}$$

or, cleaning up a bit:

$$-16\pi G T_{\mu\nu} = \bar{\Box} h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{\Box} h. \tag{31}$$

By defining a so-called trace-reversed pertubation:

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \tag{32}$$

we can give this an even more elegant form:

$$\bar{\Box}\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}. \tag{33}$$

Before we fix the rest of our degrees of freedom, we can first start solving these equations in a more general setting, and then impose gauge conditions on the solutions we find.

2.4 Vacuum Equations

In order to get a feel for how gravitational waves propagate through spacetime, it is educational to first look at the vacuum case ([5]):

$$\bar{\Box}\bar{h}_{\mu\nu} = 0. \tag{34}$$

Of course, the solutions to this equation are well know, and the subset of these is spanned by the plane waves, so let us assume that the solution is a plane wave:

$$\bar{h}_{\mu\nu} = C_{\mu\nu}e^{ik_\lambda x^\lambda}. \tag{35}$$

Note that this solution is complex, we will take the real part if we want to have the physical result. Now we can start fixing the constants by plugging in:

$$\begin{aligned} 0 &= \bar{\Box}\bar{h}_{\mu\nu} \\ &= \eta^{\rho\sigma}\partial_\rho\partial_\sigma C_{\mu\nu}e^{ik_\lambda x^\lambda} \\ &= -\eta^{\rho\sigma}k_\rho k_\sigma C_{\mu\nu}e^{ik_\lambda x^\lambda} \\ &= -k_\rho k^\rho \bar{h}_{\mu\nu}. \end{aligned} \tag{36}$$

We are not interested in the case that $\bar{h}_{\mu\nu}$ vanishes in all components, so we should have $k_\rho k^\rho = 0$, in other words: the wave vector is light-like. This means that gravitational waves propagate at the speed of light. Usually, one splits the wave vector in to a time-component and a space-vector: $k^\rho = (\omega, \mathbf{k})$, where ω is the frequency of the wave. In this terminology, the wave being light-like can be expressed as:

$$\omega^2 = \mathbf{k}^2. \tag{37}$$

So now we still have thirteen degrees of freedom left: ten free components in $C_{\mu\nu}$ and three in k^ρ. We can eliminated quite of lot of these by applying gauge conditions, starting with the harmonic gauge:

$$\begin{aligned} 0 &= \partial_\kappa h^\kappa_{\ \nu} - \frac{1}{2}\partial_\nu h \\ &= \partial_\kappa \bar{h}^\kappa_{\ \nu} \\ &= iC^\kappa_{\ \nu}k_\kappa e^{ik_\lambda x^\lambda}. \end{aligned} \tag{38}$$

The exponent does not vanish everywhere, so for (38) to hold, we must have that

$$C_{\kappa\nu}k^\kappa = 0, \tag{39}$$

which is telling us that the wave vector is orthogonal to the perturbation. Equation (39) has one free index, so it gives us four conditions on $C_{\mu\nu}$, reducing the number of free components in $C_{\mu\nu}$ to six. To fix these we can go to the transverse traceless subgauge (the name will become clear in a moment), by converting $C_{\mu\nu}$ such that:

$$C^{\mu}_{\ \mu} = 0 \tag{40}$$

and

$$C_{0\nu} = 0. \tag{41}$$

Of course, one still has to show that this conversion can actually be performed with the freedom left. This freedom consists of the harmonic gauge being invariant under translation by a set of harmonic coordinates, for ξ^μ such that $\Box\xi^\mu = 0$ we have that for the coordinate transformation $x^\mu \to x^\mu + \xi^\mu$:

$$\Box(x^\mu + \xi^\mu) = \Box x^\mu = 0. \tag{42}$$

However, showing how this can be used to transform $C_{\mu\nu}$ gives little insight at the cost of much calculating. For a complete treatment, see [5]. With condition (41) we also picked the Lorentz frame (rest frame) to construct our solution in, while the actual gauge condition is $U^\mu C_{\mu\nu} = 0$, with U^μ the four-velocity of this frame. This condition is also actually just three extra conditions, for it implies (39) for $\nu = 0$.
Counting degrees of freedom tells us that, from the sixteen components in $C_{\mu\nu}$, we had ten left by symmetry, fixed four by (39), and another four by (40) and (41), leaving us with two independent components in $C_{\mu\nu}$. Because we have used all our gauge freedom, these components must have some physical meaning. The last thing we can choose is the axis along which the wave propagates, we will choose the x^3 axis. This immediately gives us for the light-like (equation (37)) wave vector:

$$k^\mu = (\omega, 0, 0, k^3) = (\omega, 0, 0, \pm\omega). \tag{43}$$

For $C_{\mu\nu}$ we can use that $C_{0\nu} = 0$ to conclude that, because

$$0 = k^\mu C_{\mu\nu} = -\omega C_{0\nu} + k^3 C_{3\nu} = \pm\omega C_{3\nu}, \tag{44}$$

we have:

$$C_{3\nu} = 0. \tag{45}$$

Combining this with the tracelessness and symmetry of $C_{\mu\nu}$ we get for its matrix form:

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C_{11} & C_{12} & 0 \\ 0 & C_{12} & -C_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{46}$$

So we have found that we can characterize gravitational wave traveling along the x^3 direction by C_{11}, C_{12} and ω. Note that $\bar{h}_{\mu\nu}$ is traceless, and we thus have $\bar{h}_{\mu\nu} = h_{\mu\nu}$ in this gauge. By now the name transverse traceless should also be clear: the perturbation is traceless, and the perturbations are perpendicular to the direction of propagation, thus transverse.
Incidentally, this procedure for fixing a gauge also carries with it a convenient method for converting an arbitrary tensor $B_{\mu\nu}$ to this transverse-traceless frame

of reference. First, we pick just the spatial part of the tensor, B_{ij}, and then we project the tensor on to the subspace orthogonal to the spatial direction of propagation of the gravitational wave, making it transverse. We do this defining a projection operator P_{ij}, which leaves the transverse part of the tensor intact, and sets the rest to zero. Leaving intact means using the identity operator δ_{ij}, and setting to zero is done by selecting the part of the tensor in the direction of the wave by contracting with $\hat{\mathbf{k}}_j$, and subtracting it from the tensor by multiplying with $\hat{\mathbf{k}}_i$, where the $\hat{\mathbf{k}}_i$ is the unit vector in the direction of propagation. So:

$$P_{ij} = \delta_{ij} - \hat{\mathbf{k}}_i\hat{\mathbf{k}}_j. \tag{47}$$

This is indeed a projection operator (when talking about this operator we will always ignore the upper and lower index convention, and just sum over identical indices):

$$P_{ij}P_{jl} = \delta_{ij}\delta_{jl} - \delta_{ij}\hat{\mathbf{k}}_j\hat{\mathbf{k}}_l - \hat{\mathbf{k}}_i\hat{\mathbf{k}}_j\delta_{jl} + \hat{\mathbf{k}}_i\hat{\mathbf{k}}_j\hat{\mathbf{k}}_j\hat{\mathbf{k}}_l = \delta_{il} - 2\hat{\mathbf{k}}_i\hat{\mathbf{k}}_l + \hat{\mathbf{k}}_j\hat{\mathbf{k}}_l = P_{il}, \tag{48}$$

where we used that $\hat{\mathbf{k}}_j\hat{\mathbf{k}}_j = 1$. It projects on the subspace orthogonal to the direction of propagation:

$$P_{ij}\mathbf{k}_i = \delta_{ij}\mathbf{k}_i - \hat{\mathbf{k}}_i\hat{\mathbf{k}}_j\mathbf{k}_i = \mathbf{k}_j - \mathbf{k}_j = 0. \tag{49}$$

We can then make the tensor transverse by:

$$B^{\mathrm{T}}_{ij} = P_{il}P_{jk}B_{lk}, \tag{50}$$

which satisfies $B^{\mathrm{T}}_{ij}\mathbf{k}_i = 0$ by (49). To finish making the tensor transverse-traceless, we need to make sure the trace vanishes, so we subtract the trace of the projected tensor B^{T}_{ii} multiplied by the identity on the orthogonal subspace, P_{ij}, divided by its trace, 2:

$$B^{\mathrm{TT}}_{ij} = B^{\mathrm{T}}_{ij} - \frac{1}{2}P_{ij}B^{\mathrm{T}}_{mm} = (P_{il}P_{jk} - \frac{1}{2}P_{ij}P_{ml}P_{mk})B_{lk} = (P_{il}P_{jk} - \frac{1}{2}P_{ij}P_{lk})B_{lk}. \tag{51}$$

This will be extremely useful in finding the components of the energy-momentum tensor that generate gravitational waves.

2.5 Gravitational Waves in Empty Space

We set out to solve the vacuum equations to see how gravitational waves propagate through empty space, so let us have a look at what the waves do with test particles. We know, from the discussion above, that we have two independent components in $C_{\mu\nu}$. We now claim that these independent components represent two independent polarizations, and for convenience and according to convention, rename them: $C_+ := C_{11}$ and $C_\times := C_{12}$, and treat them separately.

For both polarizations, we will consider a ring of particles (radius L) lying in the x^1, x^2-plane, centered around some point O^μ, and see how the unperturbed distance to the point is affected by the waves. The unperturbed distance is of course just the length of a straight line, as far as purely spatial separation is concerned.

Consider the +-polarization first: assume that our metric is of the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \hat{h}_{\mu\nu}, \tag{52}$$

with $\hat{h}_{\mu\nu}$ given by:

$$\hat{h}_{\mu\nu}(x^\mu) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C_+ & 0 & 0 \\ 0 & 0 & -C_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{ik_\lambda x^\lambda}. \tag{53}$$

Making the ring of particles idea more formal, we can parametrize by the angle with respect to the x^1-axis, θ, to get:

$$q^\mu(\theta) = O^\mu + (0, L\cos\theta, L\sin\theta, 0)^\mu. \tag{54}$$

So the unit tangent vector to the straight line from O^μ to a point $q^\mu(\theta)$ on the ring is given by:

$$n^\mu(\theta) = (0, \cos\theta, \sin\theta, 0)^\mu, \tag{55}$$

and for $\lambda \in [0, L]$ we can parametrize the straight line by:

$$x^\mu(\lambda, \theta) = O^\mu + \lambda n^\mu. \tag{56}$$

In General Relativity the length of a path is given by:

$$s = \int_0^b \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda, \tag{57}$$

with λ an affine parameter and the path beginning at $\lambda = 0$ and ending at $\lambda = b$. Plugging in (52) and (55), we get for the perturbed distance L':

$$\begin{aligned} L'_+ &= \int_0^L \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda \\ &= \int_0^L \sqrt{g_{ij}(x^\mu(\lambda,\theta))n^i(\theta)n^j(\theta)}d\lambda \\ &= \int_0^L \sqrt{\eta_{ij}n^in^j + \hat{h}_{ij}(x^\mu(\lambda,\theta))n^i(\theta)n^j(\theta)}d\lambda \\ &= \int_0^L \sqrt{1 + \hat{h}_{ij}(x^\mu(\lambda,\theta))n^i(\theta)n^j(\theta)}d\lambda \\ &= \int_0^L (1 + \frac{1}{2}\hat{h}_{ij}(x^\mu(\lambda,\theta))n^i(\theta)n^j(\theta))d\lambda \\ &= L + \frac{1}{2}\int_0^L \hat{h}_{ij}(x^\mu(\lambda,\theta))n^i(\theta)n^j(\theta)d\lambda, \end{aligned} \tag{58}$$

where we used the fact that n^μ is of unit length, and $\hat{h}_{ij} \ll 1$ allowed us to expand the square root around 1. We can explicitly calculate the perturbation $\delta L := L' - L$, by plugging in (53) and (43) with $k_3 = \omega$:

$$\begin{aligned} \delta L_+ &= \frac{1}{2}\int_0^L C_+(\cos^2\theta - \sin^2\theta)e^{ik_\alpha(O^\alpha + \lambda n^\alpha)}d\lambda \\ &= \frac{1}{2}\int_0^L C_+\cos(2\theta)e^{i\omega(-O^0+O^4)}d\lambda \\ &= \frac{LC_+}{2}\cos(2\theta)e^{i\omega(-O^0+O^4)}. \end{aligned} \tag{59}$$

Finally, we can give a description of the perturbed ring by replacing L with $L'(O^\mu, \theta)$ in (54), since the perturbation is that of the straight line path:

$$\begin{aligned} q'^{\mu}_{+}(O^\mu, \theta) &= O^\mu + L'(0, \cos\theta, \sin\theta, 0)^\mu \\ &= q^\mu(O^\mu, \theta) + \frac{1}{2} L C_{+} e^{i\omega(-O^0+O^4)} \cos(2\theta) n^\mu(\theta). \end{aligned} \tag{60}$$

Fixing $O^4 = 0$ and letting O^0 run, we can see what a $+$-polarized gravitational wave does with a slice of spacetime by plotting the real part of this:

$$\hat{q}_{+}(O^0, \theta) = L(1 + \frac{1}{2} C_{+} \cos(\omega O^0) \cos(2\theta) (\cos\theta, \sin\theta), \tag{61}$$

for different values of O^0. The result is shown in figure 2. We can treat the

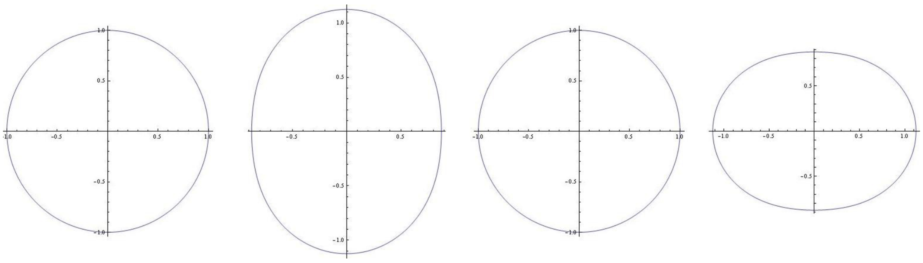

Figure 2: Plus polarisation. The displacement of the test particles in the ring is grossly exagurated.

$\times$-polarization in the same manner, up to equation (58). Here we plug in

$$\hat{h}_{\mu\nu}(x^\mu) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & C_\times & 0 \\ 0 & C_\times & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{ik_\lambda x^\lambda}. \tag{62}$$

to get for the perturbation in the distance:

$$\begin{aligned} \delta L_\times &= \frac{1}{2} \int_0^L C_\times (2\cos\theta \sin\theta) e^{ik_\alpha(O^\alpha + \lambda n^\alpha)} d\lambda \\ &= \frac{1}{2} \int_0^L C_\times \sin(2\theta) e^{i\omega(-O^0+O^4)} d\lambda \\ &= \frac{1}{2} L C_\times \sin(2\theta) e^{i\omega(-O^0+O^4)}. \end{aligned} \tag{63}$$

Again fixing $O^4 = 0$ and plotting

$$\hat{q}_\times(O^0, \theta) = L(1 + \frac{1}{2} C_\times \cos(\omega O^0) \sin(2\theta) (\cos\theta, \sin\theta) \tag{64}$$

for different values of O^0 gives us a picture of what a $\times$-polarized gravitational wave does with a slice of spacetime. The plot is shown in figure 3. Now we know what a gravitational wave does with spacetime, and thus have some more intuition about what they are, we can proceed by looking at where they come from.

2.6 Einstein Equations in Matter

The linearized Einstein equations with a non-zero energy-momentum tensor, in harmonic gauge, are given by equation (33):

$$\bar{\Box}\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}. \tag{65}$$

To get some feeling for what kind of sources cause gravitational waves we need to look at the solution to these equations. However, the calculations involved in solving (33) are somewhat lengthy, so, because we are only really interested in the end-result, we will sketch the derivation as done in [5] instead of going into to much detail.

One of the physicist's favorite methods for solving an equation like (33), with a differential operator acting on one side and a source on the other, is using a Green's function. In this case we are only interested in what effect a source that lies inside the past light cone of a certain point has. The retarded Green's function $G(x^\sigma - y^\sigma)$ for this particular differential operator is given by:

$$G(x^\sigma - y^\sigma) = -\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\delta(|\mathbf{x}-\mathbf{y}| - (\mathbf{x^0} - \mathbf{y^0}))\theta(\mathbf{x^0} - \mathbf{y^0})), \tag{66}$$

where θ denotes the Heaviside function, which serves to select the cases $x^0 \geq y^0$ ($\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1$ for $x \geq 0$). We can now write down a general solution to (33):

$$\begin{aligned} \bar{h}_{\mu\nu}(x^\sigma) &= -16\pi G \int G(x^\sigma - y^\sigma) T_{\mu\nu}(y^\sigma) d^4y \\ &= 4G \int \frac{1}{|\mathbf{x}-\mathbf{y}|} T_{\mu\nu}(x^0 - |\mathbf{x}-\mathbf{y}|, \mathbf{y}) d^3\mathbf{y}, \end{aligned} \tag{67}$$

where in the second identity we have just integrated over y^0, and used the delta function. Upon restoring c, the 0-argument in $T_{\mu\nu}$ becomes $x^0 - |\mathbf{x}-\mathbf{y}|/c$, so we see this is the time at which a wave reaching x^σ would have been emitted. This time is referred to as the retarded time, t_r.
To turn this general solution into something we can give some more physical interpretation to, we need to make some assumptions about our energy-momentum tensor. Here we assume it is an isolated, far away and slowly moving source. Isolated and far away together translate into treating the source as centered around some point at a spatial distance R, with the difference in distance between different parts of the source is a most δR, with $\delta R \ll R$. Slowly moving

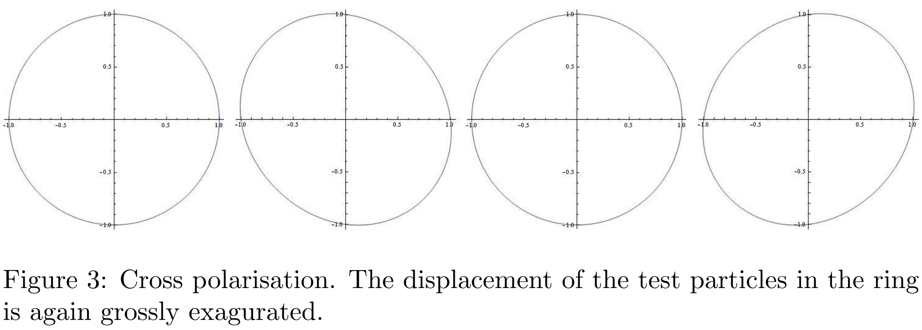

Figure 3: Cross polarisation. The displacement of the test particles in the ring is again grossly exagurated.

means that the frequency ω of the waves emitted will be such that $\delta R \ll \omega^{-1}$. Looking at the Fourier transform with respect to the time dependence in our general solution gives:

$$\tilde{\bar{h}}_{\mu\nu}(\omega, \mathbf{x}) = 4G \int e^{i\omega|\mathbf{x}-\mathbf{y}|} \frac{\tilde{T}_{\mu\nu}(\omega, \mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d^3\mathbf{y}. \tag{68}$$

Our assumptions imply that $e^{i\omega|\mathbf{x}-\mathbf{y}|}/|\mathbf{x}-\mathbf{y}|$ can replaced by $e^{i\omega R}/R$, thus neglecting the $\mathbf{y}$ dependence, and we can bring this term outside the integral. We can now use the harmonic gauge condition to limit the number of components of $\tilde{\bar{h}}_{\mu\nu}$ we need to compute. In frequency space the gauge condition is $-i\omega\tilde{\bar{h}}^{0\nu} = \partial_i\tilde{\bar{h}}^{i\nu}$, which implies $\tilde{\bar{h}}^{00} = \frac{i}{\omega}\partial_i\tilde{\bar{h}}^{i0} = \partial_i\partial_j\tilde{\bar{h}}^{ij}$, so we can limit ourselves to computing the spacelike components of $\tilde{\bar{h}}_{\mu\nu}$. This means that we have to integrate the spacelike components of $\tilde{T}_{\mu\nu}$. This can be simplified by using the conservation of energy equation up to first order in $h_{\mu\nu}$: $\partial_\mu T^{\mu\nu}$, which in frequency space becomes:

$$-\partial_k\tilde{T}^{k\mu} = i\omega\tilde{T}^{0\mu}. \tag{69}$$

Repeatedly integrating (68) by parts in reverse (to get the $\partial_k\tilde{T}^{k\mu}$ in) and noting that boundary terms vanish by the assumption that the source is isolated, we can express the integral over the $\tilde{T}^{ij}$ in terms of an integral over $\tilde{T}^{00}$:

$$\int \tilde{T}^{ij}(\omega, \mathbf{y}) d^3\mathbf{y} = -\frac{\omega^2}{2} \int y^i y^j \tilde{T}^{00}(\omega, \mathbf{y}) d^3\mathbf{y}. \tag{70}$$

If we now define the quadropole moment tensor of the energy density of the source to be:

$$q_{ij}(y^0) = 3 \int y^i y^j T^{00}(y^0, \mathbf{y}) d^3\mathbf{y}, \tag{71}$$

we get, after inverting the Fourier transform:

$$\bar{h}_{ij}(x^\sigma) = \frac{2G}{3R} \frac{\partial^2 q_{ij}}{\partial y^{0\,2}}(t_r). \tag{72}$$

So we see that a source with a non-vanishing second derivative of its quadrupole moment tensor of the energy density emits gravitational waves. The quadrupole moment tensor of energy density is a measure for the shape of the source, it measures how the energy density (thus the mass) is distributed around the center of the source. The second derivative of this then measures any non-uniformities in the change of the shape over time, and it is these non-uniformities that generate gravitational waves.
This means that for example the spherically symmetric collapse of a star will not generate any gravitational waves (the spherical symmetry $\partial_0^2 T^{00}(y^0, \mathbf{y}) = -\partial_0^2 T^{00}(y^0, -\mathbf{y})$, together with the quadratic dependence on the distance in (71) implies $\partial_0^2 q_{ij} = 0$). However, binary stars will emit gravitational radiation. This is an important example of an astrophysical source, because the strongest indication that gravitational waves do indeed exist is the consistency between the energy loss (observable through a decrease in the period of the orbit) of the binary pulsar PSR B1913+16 and the predicted energy loss due to gravitational radiation. For a treatment of gravitational waves from binary stars in general and the energy loss of PSR B1913+16 in particular see [8].
Later in this text we will see how gravitational waves are generated by bubble collisions during phase transitions.

3 Friedmann-Lemaître-Robertson-Walker

3.1 Einstein Equations

In this section we will briefly review the basic notions of Friedmann-Lemaître-Robertson-Walker cosmology. The reader is assumed to be familiar with this model, but we will need the Friedmann equations themselves as well as some results obtained in deriving them later on in the text. In FLRW cosmology one makes the following Ansatz for the metric:

$$g_{\mu\nu} = a(\eta)^2 \eta_{\mu\nu}, \tag{73}$$

where $\eta_{\mu\nu}$ is just the Minkowski metric, $a(\eta)$ is called the scale factor and is a function of the conformal time η, which is related to the cosmic time t by $a d\eta = dt$. This Ansatz assumes that the universe is isotropic, homogeneous and spatially flat. We proceed by computing the Einstein tensor for this metric. Observe that the inverse metric is given by:

$$g^{\mu\nu} = \frac{1}{a^2}\eta^{\mu\nu} \tag{74}$$

The Christoffel symbols are given by:

$$\begin{aligned} \Gamma^{\rho}_{\mu\nu} &= \frac{1}{2} g^{\rho\beta}(\partial_\nu g_{\beta\mu} + \partial_\mu g_{\beta\nu} - \partial_\beta g_{\mu\nu}) \\ &= \frac{a'}{a}(\delta^0_\mu \delta^\rho_\nu + \delta^0_\nu \delta^\rho_\mu - \delta^\rho_0 \eta_{\mu\nu}), \end{aligned} \tag{75}$$

where a' denotes the derivative of a with respect to conformal time, and the main observations in the calculations are $\partial_\lambda g_{\kappa\rho} = \delta^0_\lambda \eta_{\kappa\rho} a a'$ and $\eta^{\kappa\lambda}\eta_{\lambda\rho} = \delta^\kappa_\rho$. On to the Ricci tensor, which is given by:

$$\begin{aligned} R_{\mu\nu} &= \partial_\alpha \Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\lambda\alpha}\Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} - \Gamma^\alpha_{\lambda\nu}\Gamma^\lambda_{\mu\alpha} \\ &= \left(\frac{a''}{a} - \left(\frac{a'}{a}\right)^2\right)\left(-2\delta^0_\mu\delta^0_\nu + \eta_{\mu\nu}\right) + \left(\frac{a'}{a}\right)^2)\left(2\delta^0_\mu\delta^0_\nu + 2\eta_{\mu\nu}\right) \\ &= \left(\frac{a''}{a^3} - 2\left(\frac{a'}{a^2}\right)^2\right)\left(-2\delta^0_\mu\delta^0_\nu a^2 + g_{\mu\nu}\right) + 3\left(\frac{a'}{a^2}\right)^2 g_{\mu\nu}. \end{aligned} \tag{76}$$

We can rewrite this in terms of the Hubble parameter $H(t) = \frac{\dot{a}}{a} = \frac{a'}{a^2}$ (with $\dot{a} = \frac{da}{dt}$), and its derivative $\dot{H} = \frac{a''}{a^3} - 2\frac{a'^2}{a^4} = \frac{H'}{a}$:

$$R_{\mu\nu} = \dot{H}\left(-2\delta^0_\mu\delta^0_\nu a^2 + g_{\mu\nu}\right) + 3H^2 g_{\mu\nu}. \tag{77}$$

Taking the trace yields the Ricci scalar:

$$R = g^{\mu\nu}R_{\mu\nu} = 6\dot{H} + 12H^2, \tag{78}$$

and we get for the Einstein tensor:

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \\ &= \dot{H}\left(-2\delta^0_\mu\delta^0_\nu a^2 - 2g_{\mu\nu}\right) - 3H^2 g_{\mu\nu}. \end{aligned} \tag{79}$$

We then get for the Einstein equations with cosmological constant Λ:

$$\begin{aligned} 8\pi G T_{\mu\nu} &= G_{\mu\nu} + \Lambda g_{\mu\nu} \\ &= \dot{H}\left(-2\delta^0_\mu\delta^0_\nu a^2 - 2g_{\mu\nu}\right) - (3H^2 - \Lambda) g_{\mu\nu}. \end{aligned} \tag{80}$$

3.2 Stress-Energy Tensor

In order to solve the Einstein equations (80), we need to make some assumptions about the stress-energy tensor. We assume it to be translational and rotational invariant, which implies it to be that of a perfect fluid ([17]). We thus assume it to be of the form:

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}, \tag{81}$$

where ρ is the energy density, p is the pressure and U^μ is the four-velocity of whatever it is we are constructing the stress-energy tensor for ($U^\mu = (\frac{1}{a}, 0, 0, 0)$ in the rest frame). Next, we assume there is some relation between the pressure and the energy density, $w = \frac{p}{\rho}$, this equation is called the equation of state. For a dust, where we assume the particles do not interact, we have $w = 0$. This corresponds to the so called matter era. For radiation we take $w = \frac{1}{3}$, this corresponds to the radiation era.
We thus get, in the rest frame of the fluid, for the stress-energy tensor in the radiation era:

$$T_{\mu\nu} = \mathrm{diag}(\rho, p, p, p)a^2. \tag{82}$$

Note that our assumption that the universe is isotropic and homegeneous implies that ρ and p are functions of time alone.
We can already extract quite a lot of information from these assumptions, by conservation of energy we have (again in the fluid's restframe):

$$\begin{aligned} 0 &= \nabla^\mu T_{\mu\nu} \\ 0 = \nabla^\mu T_{\mu 0} &= \dot{\rho} + 3H(\rho + p) \qquad (83) \\ &= \dot{\rho} + 3\frac{\dot{a}}{a}(1 + w)\rho \\ &= (a^{3(1+w)}\rho)^\cdot, \qquad (84) \end{aligned}$$

where we just considered the $\nu = 0$ equation since the rest is trivially satisfied.
From this we get for ρ:

$$\rho = \frac{\rho_0}{a^{3(1+w)}}, \tag{85}$$

and for H:

$$H = \frac{2}{3(1 + w)t}. \tag{86}$$

3.3 Friedmann Equations

Now that we have a form for the energy-momentum tensor, we can write down explicit equations for $a(t)$ in terms of the pressure and energy density.
From the 00-component of the Einstein equations (80) we get:

$$\begin{aligned} 8\pi T_{00} &= G_{00} + \Lambda g_{00} \\ 8\pi\rho a^2 &= \dot{H}(-2a^2 - 2(-a^2)) + (3H^2 - \Lambda)a^2 - \Lambda a^2 \\ 8\pi\rho a^2 &= 3\dot{a}^2 - 2\Lambda a^2. \qquad (87) \end{aligned}$$

This equation together with (84) are called the Friedmann equations for a spatially flat universe. The rest of the components of the Einstein equations give us just the same equation as the conservation law (83). Setting Λ to zero, or absorbing it into the energy density ρ, this equation yields for H:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho, \tag{88}$$

thus allowing us to express the Hubble factor in terms of the energy density.

4 Gravitational Waves on a FLRW-background

4.1 Linearized Einstein Equations on FLRW-background

Being interested in cosmological sources for gravitational waves also means having to care about how these waves propagate toward us over a very long period of time. The FLRW cosmology tells us the universe expanded during its history, and this can be taken into account by doing our linearized General Relativity on a FLRW background metric, we will do this guided by [17]. We will go through very much the same procedure as for the linearized theory on Minkowsky background, but this time the algebra involved is much less trivial, since not all derivatives of the unperturbed metric are zero. In this chapter a bar on top of a quantity will denote the unperturbed value. Our metric is given by (in terms of cosmic time instead of conformal time):

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{89}$$

where $\bar{g}_{\mu\nu}$ denotes the unperturbed spatially flat FRLW-metric, in terms of cosmic time:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a(t)^2\delta_{ij}dx^i dx^j, \tag{90}$$

and $h_{\mu\nu}$ is again a small, symmetric perturbation. To proceed with calculating the Christoffel symbols, we also need the perturbation of the inverse metric:

$$h^{\mu\nu} = g^{\mu\nu} - \bar{g}^{\mu\nu} = \bar{g}^{\mu\nu} - \bar{g}^{\mu\lambda}\bar{g}^{\mu\kappa}h_{\mu\nu} - \bar{g}^{\mu\nu} = -\bar{g}^{\mu\lambda}\bar{g}^{\mu\kappa}h_{\lambda\kappa}. \tag{91}$$

Instead of just bluntly starting the calculations for the Christoffel symbols, it pays to think a bit about what we should get. The formula for the Christoffel symbols is (9):

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}).$$

We know that the inverse metric part consist of an unperturbed and a perturbed part, and so does the term in brackets. Working out the product will then give us four terms: one fully unperturbed term, one with an unperturbed inverse metric and a perturbed bracket factor, one the other way around and one term with only perturbation factors. We drop the latter, since we are only interested in terms linear in the perturbations. The first term of the four we already know from equation (75), it is just the unperturbed Christoffel symbols, denoted by $\bar{\Gamma}^\rho_{\mu\nu}$. We can rewrite the term with the unperturbed bracket factor in terms of the unperturbed Christoffel symbols by using equation (91):

$$\begin{aligned} \frac{1}{2}h^{\rho\sigma}(\partial_\mu \bar{g}_{\sigma\nu} + \partial_\nu \bar{g}_{\mu\sigma} - \partial_\sigma \bar{g}_{\mu\nu}) &= -\frac{1}{2}\bar{g}^{\rho\lambda}\bar{g}^{\sigma\kappa}h_{\lambda\kappa}(\partial_\mu \bar{g}_{\sigma\nu} + \partial_\nu \bar{g}_{\mu\sigma} - \partial_\sigma \bar{g}_{\mu\nu}) \\ &= -\bar{g}^{\rho\lambda}h_{\lambda\kappa}\bar{\Gamma}^\kappa_{\mu\nu}. \end{aligned} \tag{92}$$

This gives us for the perturbation of the Christoffel symbols:

$$\delta\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \bar{\Gamma}^\rho_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\lambda}(-2h_{\lambda\kappa}\bar{\Gamma}^\kappa_{\mu\nu} + \partial_\mu h_{\sigma\nu} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}). \tag{93}$$

Note that the unperturbed non-zero components of the Christoffel symbols (75) in terms of cosmic time become:

$$\bar{\Gamma}^i_{0j} = \frac{\dot{a}}{a}\delta^i_j,\ \bar{\Gamma}^0_{ij} = a\dot{a}\delta_{ij}. \tag{94}$$

Plugging this into (93) gives us for the components of the perturbation of Christoffel symbols (from here on we plug in the explicit form for the metric, so the difference between upper and lower indices is from here on merely typographical):

$$\begin{aligned}
\delta\Gamma^i_{jk} &= \frac{1}{2a^2}\delta^{in}(-2a\dot{a}h_{n0}\delta_{jk} + \partial_j h_{nk} + \partial_k h_{nj} - \partial_n h_{jk}) \\
&= \frac{1}{2a^2}(-2a\dot{a}h_{i0}\delta_{jk} + \partial_j h_{ik} + \partial_k h_{ij} - \partial_i h_{jk}) && (95) \\
\delta\Gamma^i_{j0} &= \frac{1}{2a^2}\delta^{in}(-\frac{2\dot{a}}{a}h_{nm}\delta^m_j + \partial_0 h_{jn} + \partial_j h_{0n} - \partial_n h_{0j}) \\
&= \frac{1}{2a^2}(-\frac{2\dot{a}}{a}h_{ij} + \partial_0 h_{ij} + \partial_j h_{0i} - \partial_i h_{0j}) && (96) \\
\delta\Gamma^0_{ij} &= -\frac{1}{2}(-2a\dot{a}h_{00}\delta_{ij} + \partial_j h_{0i} + \partial_i h_{0j} - \partial_0 h_{ij}) && (97) \\
\delta\Gamma^i_{00} &= \frac{1}{2a^2}\delta^{in}(\partial_0 h_{0n} + \partial_0 h_{0n} - \partial_n h00) \\
&= \frac{1}{2a^2}(2\partial_0 h_{i0} - \partial_i h_{00}) && (98) \\
\delta\Gamma^0_{i0} &= -\frac{1}{2}(-2\frac{\dot{a}}{a}h_{0n}\delta_{in} + \partial_0 h_{0i} + \partial_i h_{00} - \partial_0 h_{i0}) \\
&= \frac{\dot{a}}{a}h_{i0} - \frac{1}{2}\partial_i h_{00} && (99) \\
\delta\Gamma^0_{00} &= -\frac{1}{2}(\partial_0 h_{00} + \partial_0 h_{00} - \partial_0 h_{00}) \\
&= -\frac{1}{2}\partial_0 h_{00} && (100)
\end{aligned}$$

The next observation that can save us quite a lot of calculations is that we are only really interested in the perturbation to the Einstein equations, since we know the unperturbed equations from (80). In terms of the unperturbed Christoffel symbols and up to first order in their perturbations, the perturbation of the Ricci tensor is given by:

$$\delta R_{\mu\nu} = \partial_\lambda \delta\Gamma^\lambda_{\mu\nu} - \partial_\nu \delta\Gamma^\lambda_{\mu\lambda} + \delta\Gamma^\lambda_{\kappa\lambda}\bar{\Gamma}^\kappa_{\mu\nu} + \delta\Gamma^\kappa_{\mu\nu}\bar{\Gamma}^\lambda_{\kappa\lambda} - \delta\Gamma^\kappa_{\mu\lambda}\bar{\Gamma}^\lambda_{\kappa\nu} - \delta\Gamma^\lambda_{\kappa\nu}\bar{\Gamma}^\kappa_{\mu\lambda} \quad (101)$$

Which gives us for the components (for an explicit calculation, see Appendix A):

$$\begin{aligned}
\delta R_{ij} &= \frac{1}{2}\partial_j\partial_i h_{00} + (\dot{a}^2 + a\ddot{a})h_{00}\delta_{ij} + \frac{1}{2}a\dot{a}\dot{h}_{00}\delta_{ij} + \frac{\dot{a}}{2a}(\dot{h}_{kk}\delta_{ij} - \dot{h}_{ij}) + \frac{1}{2}\ddot{h}_{ij} \\
&+ \frac{1}{2a^2}(\partial_k\partial_i h_{jk} + \partial_k\partial_j h_{ij} - \nabla^2 h_{ij} - \partial_j\partial_i h_{kk}) + \frac{\dot{a}^2}{a^2}(-h_{kk}\delta_{ij} + 2h_{ij}) \\
&- \frac{\dot{a}}{a}\partial_k h_{k0}\delta_{ij} - \frac{1}{2}(\partial_j\dot{h}_{0i} + \partial_i\dot{h}_{0j}) + \dot{a}^2 h_{00}\delta_{ij} - \frac{3\dot{a}}{2a}(\partial_j h_{0i} + \partial_i h_{0j}) \\
&- \frac{\dot{a}}{2a}(\partial_j h_{0i} + \partial_i h_{0j}), && (102)
\end{aligned}$$

$$\begin{aligned}
\delta R_{0j} &= \delta R_{j0} = -\frac{\dot a}{a}\partial_j h_{00} + \frac{1}{2a^2}(\partial_i\partial_j h_{i0} - \nabla^2 h_{j0}) + (\frac{\ddot a}{a} + \frac{2\dot a^2}{a^2})h_{0j} \\
&\quad -\frac{1}{2}\partial_0(\frac{1}{a^2}(\partial_j h_{ii} - \partial_i h_{ij})), \qquad (103)\\
\delta R_{00} &= -\frac{1}{2a}\nabla^2 h_{00} - \frac{3\dot a}{2a}\dot h_{00} + \frac{1}{a^2}\partial_i \dot h_{0i} \\
&\quad -\frac{1}{2a^2}(\ddot h_{ii} - \frac{2\dot a}{a}\dot h_{ii} + 2(\frac{\dot a^2}{a^2} - \frac{\ddot a}{a})h_{ii}). \qquad (104)
\end{aligned}$$

Now, we rewrite the Einstein equations a bit in order to save us the agony of going through taking the trace of the Ricci tensor:

$$\begin{aligned}
8\pi G T_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^\kappa_\kappa \\
8\pi G T^\alpha_\alpha &= -R^\alpha_\alpha \\
8\pi G(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\alpha_\alpha) &= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^\kappa_\kappa + \frac{1}{2}R^\alpha_\alpha g_{\mu\nu} \\
8\pi G(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\alpha_\alpha) &= R_{\mu\nu}. \qquad (105)
\end{aligned}$$

The tensor on the left hand side of this equation is defined to be the source tensor:

$$S_{\mu\nu} := T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\alpha_\alpha. \qquad (106)$$

Since we are interested in the perturbation of the Einstein equations, we want to know what the perturbation in the source tensor is up to first order in perturbations in both the stress-energy tensor and the metric:

$$\delta S_{\mu\nu} = \delta T_{\mu\nu} - \frac{1}{2}\bar g_{\mu\nu}\delta T^\alpha_\alpha - \frac{1}{2}h_{\mu\nu}\bar T^\alpha_\alpha. \qquad (107)$$

In the end, we want to have an equation that relates the perturbation in the stress-energy tensor to a perturbation in the metric, in terms of that perturbation and the scale factor (and its derivatives) alone. We thus want to get rid of the unperturbed stress-energy tensor. Using (81) we can express $T_{\mu\nu}$ in terms of energy density and pressure, which in turn can be expressed in terms of the scale factor using the Friedmann equation (87) (setting $\Lambda = 0$):

$$\bar\rho = \frac{3}{8\pi G}\frac{\dot a^2}{a^2}, \qquad (108)$$

and thus by the conservation law (83) for $\bar p$:

$$\begin{aligned}
\bar p &= -\frac{a}{3\dot a}\dot{\bar\rho} - \bar\rho \\
&= -\frac{3}{8\pi G}\left(\frac{a}{\dot a}(\frac{2\dot a\ddot a}{a^2} - 2\frac{\dot a^3}{a^3}) + \frac{\dot a^2}{a^2}\right) \\
&= -\frac{1}{8\pi G}\left(\frac{2\ddot a}{a} + \frac{\dot a^2}{a^2}\right). \qquad (109)
\end{aligned}$$

This gives us for the trace of $\bar T_{\mu\nu}$:

$$\bar T^\alpha_\alpha = -\bar\rho + 3\bar p = -\frac{3}{4\pi G}\left(\frac{\ddot a}{a} + \frac{\dot a^2}{a^2}\right), \qquad (110)$$

and we get for the components of the perturbation in the source tensor (equation 107):

$$\delta S_{ij} = \delta T_{ij} - \frac{a^2}{2}\delta_{ij}\delta T^{\alpha}_{\ \alpha} + \frac{3}{4\pi G}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) h_{ij} \tag{111}$$

$$\delta S_{0j} = \delta T_{0j} + \frac{3}{4\pi G}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) h_{0j} \tag{112}$$

$$\delta S_{00} = \delta T_{00} + \frac{1}{2}\delta T^{\alpha}_{\ \alpha} + \frac{3}{4\pi G}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) h_{00}. \tag{113}$$

So we finally get for the linearized Einstein equations on a Friedmann-Lemaitre-Robertson-Walker background metric (moving all $h_{\mu\nu}$-terms to the right hand side):

$$\begin{aligned}
8\pi G\left(\delta T_{ij} - \frac{a^2}{2}\delta_{ij}\delta T^{\alpha}_{\ \alpha}\right) &= \frac{1}{2}\partial_j\partial_i h_{00} + (\dot{a}^2 + a\ddot{a})h_{00}\delta_{ij} + \frac{\dot{a}}{2a}(\dot{h}_{kk}\delta_{ij} - \dot{h}_{ij}) \\
&\quad + \frac{1}{2a^2}(\partial_k\partial_i h_{jk} + \partial_k\partial_j h_{ij} - \partial_i\partial_i h_{ij} - \partial_j\partial_i h_{kk}) \\
&\quad + \frac{\dot{a}^2}{a^2}(-h_{kk}\delta_{ij} - h_{ij}) - \frac{1}{2}(\partial_j\dot{h}_{0i} + \partial_i\dot{h}_{0j}) - 3\frac{\ddot{a}}{a}h_{ij} \\
&\quad - \frac{3\dot{a}}{2a}(\partial_j h_{0i} + \partial_i h_{0j}) - \frac{\dot{a}}{2a}(\partial_j h_{0i} + \partial_i h_{0j}) \\
&\quad + \frac{1}{2}a\dot{a}\dot{h}_{00}\delta_{ij} + \dot{a}^2 h_{00}\delta_{ij} + \frac{1}{2}\ddot{h}_{ij} - \frac{\dot{a}}{a}\partial_k h_{k0}\delta_{ij} \qquad (114) \\
8\pi G\delta T_{0j} &= -\frac{\dot{a}}{a}\partial_j h_{00} + \frac{1}{2a^2}(\partial_i\partial_j h_{i0} - \partial_i\partial_i h_{j0}) \\
&\quad - \frac{1}{2}\partial_0\left(\frac{1}{a^2}(\partial_j h_{ii} - \partial_i h_{ij})\right) - \left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)h_{0j} \qquad (115) \\
8\pi G\left(\delta T_{00} + \frac{1}{2}\delta T^{\alpha}_{\ \alpha}\right) &= -\frac{1}{2a}\partial_i\partial_i h_{00} + \frac{1}{a^2}\partial_i\dot{h}_{0i} - 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)h_{00} \\
&\quad - \frac{3\dot{a}}{2a}\dot{h}_{00} - \frac{1}{2a^2}\left(\ddot{h}_{ii} - \frac{2\dot{a}}{a}\dot{h}_{ii} + 2\left(\frac{\dot{a}^2}{a^2} - \frac{\ddot{a}}{a}\right)h_{ii}\right) \qquad (116)
\end{aligned}$$

4.2 Fixing a Gauge

The linearized Einstein equations above are horribly complicated, so it is time for us to think of a way of simplifying them. Just like in the case of a flat background, we can do this by imposing certain conditions on the perturbations, much like fixing a gauge. The subtle point here is, that even though we wil be calling it fixing a gauge, there is actually some physical freedom left we will be ignoring. The reason we can do this is that there is a very useful property of the linearized Einstein equations in this setting, called the decomposition theorem, which we will not prove here, a treatment can be found in [17]. This theorem tells us we can split the Einstein equations into a part where just scalar perturbations appear (treated in [2]), a part where just vector modes appear (these modes can be shown to attenuate very fast), and a part with just transverse traceless tensors appearing. These are tensor satisfying the following conditions for the transverse traceless gauge:

$$\partial^i h_{ij} = h_{ii} = h_{0\nu} = 0 \tag{117}$$

This simplifies the Einstein equations quite a lot:

$$
\begin{aligned}
8\pi G\left(\delta T_{ij}-\frac{a^2}{2}\delta_{ij}\delta T^\alpha_{\ \alpha}\right) &= -\frac{\dot a}{2a}\dot h_{ij}-\frac{1}{2a^2}\partial_i\partial_i h_{ij}-\frac{\dot a^2}{a^2}h_{ij} \\
&\quad -3\frac{\ddot a}{a}h_{ij}+\frac{1}{2}\ddot h_{ij} \qquad (118)\\
8\pi G\delta T_{0j} &= 0 \qquad (119)\\
8\pi G\left(\delta T_{00}+\frac{1}{2}\delta T^\alpha_{\ \alpha}\right) &= 0. \qquad (120)
\end{aligned}
$$

We can eliminate the trace term from (118) by noting that the right hand side of (118) should vanish when taking the trace over the space indices, giving:

$$
\begin{aligned}
g^{ij}\delta T_{ij} &= \frac{a^2}{2}g^{ij}\delta_{ij}\delta T^\alpha_{\ \alpha}\\
\frac{1}{a^2}\delta T_{ii} &= \frac{3}{2}\delta T^\alpha_{\ \alpha}. \qquad (121)
\end{aligned}
$$

But when we just take the trace of $\delta T_{\mu\nu}$ we get:

$$\delta T^\alpha_{\ \alpha}=-\delta T_{00}+\frac{1}{a^2}\delta T_{ii}. \qquad (122)$$

If we now elimate δT_{00} from (122) using 120 we get:

$$\frac{1}{2}\delta T^\alpha_{\ \alpha}=\frac{1}{a^2}\delta T_{ii}=\frac{3}{2}\delta T^\alpha_{\ \alpha}, \qquad (123)$$

and since zero is the only number that gives the same when multiplied by $\frac{1}{2}$ and $\frac{3}{2}$, we conclude that both $\delta T^\alpha_{\ \alpha}$ and δT_{ii} vanish. Equation (118) then becomes:

$$8\pi G\delta T_{ij}=-\frac{\dot a}{2a}\dot h_{ij}-\frac{1}{2a^2}\partial_i\partial_i h_{ij}-\frac{\dot a^2}{a^2}h_{ij}-3\frac{\ddot a}{a}h_{ij}+\frac{1}{2}\ddot h_{ij}. \qquad (124)$$

But we can still do better. First, we should take a better look at what δT_{ij} is. From its perfect fluid form (81) we get for the perturbation in the energy-momentum tensor to first order (in the fluid's rest frame):

$$
\begin{aligned}
\delta T_{ij} &= pg_{ij}-\bar p\bar g_{ij}=\bar p h_{ij}+a^2\Pi_{ij} \qquad (125)\\
&= -\frac{1}{8\pi G}\left(\frac{2\ddot a}{a}+\frac{\dot a^2}{a^2}\right)h_{ij}+a^2\Pi_{ij}, \qquad (126)
\end{aligned}
$$

where for the last identity we used (109), and we define Π_{ij} to be the tensorial perturbation in the product of the background metric with the pressure, also called the anisotropic stress. The above tells us that the part of $\delta T_{\mu\nu}$ we are interested in satisfies the same conditions as $h_{\mu\nu}$: $\partial_i\delta T^{(\mathrm{TT})}_{ij}=\delta T^{(\mathrm{TT})}_{ii}=\delta T^{(\mathrm{TT})}_{0\nu}=0$, and thus the same should hold for the part of Π_{ij} we are interested in. We call this the transverse traceless part of Π_{ij}, denoted by $\Pi^{(\mathrm{TT})}_{ij}$. Moving all terms containing $h_{\mu\nu}$ to the right hand side leaves us with:

$$8\pi Ga^2\Pi^{(\mathrm{TT})}_{ij}=-\frac{\dot a}{2a}\dot h_{ij}-\frac{1}{2a^2}\partial_i\partial_i h_{ij}-\frac{\ddot a}{a}h_{ij}+\frac{1}{2}\ddot h_{ij}. \qquad (127)$$

If we now define $\tilde{h}_{\mu\nu}$ by $h_{ij} = a^2\tilde{h}_{ij}$ (so $\dot{h}_{ij} = 2a\dot{a}\tilde{h}_{ij} + a^2\dot{\tilde{h}}_{ij}$ and $\ddot{h}_{ij} = 2\dot{a}^2\tilde{h}_{ij} + 2a\ddot{a}\tilde{h}_{ij} + 4a\dot{a}\dot{\tilde{h}}_{ij} + a^2\ddot{\tilde{h}}_{ij}$) we can rewrite this as:

$$\begin{aligned}
16\pi G a^2 \Pi_{ij}^{(\mathrm{TT})} &= -2\dot{a}^2\tilde{h}_{ij} - a\dot{a}\dot{\tilde{h}}_{ij} - \partial_i\partial_i\tilde{h}_{ij} - 2a\ddot{a}\tilde{h}_{ij} + 2\dot{a}^2\tilde{h}_{ij} \\
&\quad +2a\ddot{a}\tilde{h}_{ij} + 4a\dot{a}\dot{\tilde{h}}_{ij} + a^2\ddot{\tilde{h}}_{ij} \\
&= -\partial_i\partial_i\tilde{h}_{ij} + a^2\ddot{\tilde{h}}_{ij} + 3a\dot{a}\dot{\tilde{h}}_{ij} \qquad (128)
\end{aligned}$$

This equation is very similar to equation (33), the equation of motion for gravitational waves on a flat background. Note that $\bar{g}^{\mu\nu}\partial_\mu\partial_\nu = -\partial_0^2 + \frac{1}{a^2}\partial_i\partial_i$, so if we move the a^2 from the left hand side to the right hand side, we get:

$$16\pi G \Pi_{ij}^{(\mathrm{TT})} = -\bar{g}^{\mu\nu}\partial_\mu\partial_\nu\tilde{h}_{ij} + 3\frac{\dot{a}}{a}\dot{\tilde{h}}_{ij}, \qquad (129)$$

which looks like a wave equation with source $\Pi^{(TT)}$ and friction term $3\frac{\dot{a}}{a} = 3H$, which is sometimes called the Hubble friction, and tells us that the wave will lose amplitude due to the expansion of the universe.
The similarity is even more striking when we look at what the D'Alembertian $\Box$ becomes in this space-time (let $f : M \to \mathbf{R}$ be a test-function on the space-time manifold):

$$\Box f = \nabla^\mu\nabla_\mu f = g^{\mu\nu}\partial_\mu\partial_\nu f - g^{\mu\nu}\Gamma^\rho_{\mu\nu}\partial_\rho f. \qquad (130)$$

Using equation (94) for the Christoffel symbols and $\bar{g}^{ij} = \frac{1}{a^2}\delta^{ij}$ for the space-components of the inverse metric, we can evaluate the last term explicitly up to zeroth order in the perturbation (we would drop the higher order terms later on anyway):

$$\begin{aligned}
\bar{g}^{\mu\nu}\bar{\Gamma}^\rho_{\mu\nu}\partial_\rho f &= \frac{1}{a^2}\delta^{ij}a\dot{a}\delta_{ij}\dot{f} \\
&= \frac{3\dot{a}}{a}\dot{f}. \qquad (131)
\end{aligned}$$

We thus get for the equation of motion for gravitational waves in a Friedmann-Lemaître-Roberterson-Walker universe:

$$-16\pi G \Pi_{ij}^{(\mathrm{TT})} = \Box\tilde{h}_{ij}. \qquad (132)$$

5 Energy Density Spectrum

5.1 The Quantity of Interest

Gravitational waves are more than just an artifact that comes up when linearizing Einstein's theory of General Relativity. Even though they have never been detected, their existence is widely accepted, mainly due to the agreement in the predicted energy loss to gravitational radiation and the observed energy loss of the binary pulsar PSR B1913+16. When physicists theorize whether a certain phenomenon could be a source of gravitational waves a very important question that arises is: what is the energy density for a given wavelength today? More specifically, we will be interested in the spectrum of energy density, i.e. the energy density per frequency, from a specific source. The shape of this spectrum for a given source will make it distinguishable from other sources, and the peak of the spectrum will indicate to us at which frequency we are most likely to detect gravitational waves from that source.

5.2 The Gravitational Wave Energy-Momentum Tensor

In General Relativity, the energy-density occurs as the 00-component of the energy-momentum tensor. Our first step in deriving an expression for the energy-density of gravitational waves is thus finding the energy-momentum tensor for gravitational waves. A derivation for this in the so-called shortwave approximation is given in [14]. The derivation there, however, is done under the assumption of considering the linearized Einstein equations in vacuum. For gravitational waves in a cosmological setting this condition clearly is not satisfied, there is nothing vacuum-like about the energy-momentum tensor from (81). We will thus have to provide some additional arguments as to why the final result for the energy-momentum tensor for gravitational waves from [14] is still valid in a FLRW universe.
Let us start by imagining what would happen if we expand both sides of the Einstein equations (3) to orders in a perturbation of the metric. Letting δ^2 denote second order in the perturbation, we have:

$$\bar{G}_{\mu\nu} + \delta G_{\mu\nu} + \delta^2 G_{\mu\nu} + \text{h.o.} = 8\pi G(\bar{T}_{\mu\nu} + \delta T_{\mu\nu} + \delta^2 T_{\mu\nu} + \text{h.o.}). \tag{133}$$

Assuming we found a solution to the linear equations ($\delta G_{\mu\nu} = \delta T_{\mu\nu}$), and ignoring higher than second order, we are left with:

$$\bar{G}_{\mu\nu} + \delta^2 G_{\mu\nu} = 8\pi G(\bar{T}_{\mu\nu} + \delta^2 T_{\mu\nu}). \tag{134}$$

Recall that up to now we have neglected any non-linear effects of the perturbations on themselves, and we do not want to start doing this now. Assuming that the perturbations are small and vary over small scales compared to the background curvature, we can save ourselves from having to consider self-coupling to higher orders by averaging over a volume of space at the scale of a few wavelengths, we denote this by $\langle\cdot\rangle$ and pick a specific way of averaging later. This allows us to split $\delta^2 G_{\mu\nu}$ in a smooth part, which tells us how the gravitational waves perturb the background energy-momentum, and a fluctuating part, which tells us how the perturbations effect themselves. That is, we have:

$$\begin{aligned}\langle \bar{G}_{\mu\nu} + \delta^2 G_{\mu\nu}\rangle &= 8\pi G(\bar{T}_{\mu\nu} + T^{(\mathrm{GW})}_{\mu\nu}) \\ \bar{G}_{\mu\nu} + \langle \delta^2 G_{\mu\nu}\rangle &= 8\pi G(\bar{T}_{\mu\nu} + T^{(\mathrm{GW})}_{\mu\nu}) \\ \langle \delta^2 G_{\mu\nu}\rangle &= 8\pi G T^{(\mathrm{GW})}_{\mu\nu}. \end{aligned} \tag{135}$$

Here we identify $T^{(GW)}_{\mu\nu}$ with the energy-momentum tensor due to gravitational waves. That is, we neglect the second order effect of the gravitational waves on the background energy-momentum tensor, and see (135) as a means of calculating how much energy-momentum the gravitational waves generate. In order to evaluate $\delta^2 G_{\mu\nu}$ one needs to find the second order perturbation of the Ricci-tensor. The amount to algebra involved in this calculation is tremendous, and we will not present the derivation here. Instead we will just state the result for $\delta^2 R_{\mu\nu}$ found in [14], and take it from there. We thus start from:

$$\begin{aligned}\delta^2 R_{\mu\nu} &= \frac{1}{2}(\frac{1}{2}\bar{\nabla}_\mu h_{\alpha\beta}\bar{\nabla}_\nu h^{\alpha\beta} + h^{\alpha\beta}(\bar{\nabla}_\mu\bar{\nabla}_\nu h_{\alpha\beta} + \bar{\nabla}_\alpha\bar{\nabla}_\beta h_{\mu\nu} - \bar{\nabla}_\nu\bar{\nabla}_\beta h_{\alpha\mu} \\ &- \bar{\nabla}_\mu\bar{\nabla}_\beta h_{\nu\alpha}) + \bar{\nabla}^\beta h_\nu^{\ \alpha}(\bar{\nabla}_\beta h_{\mu\alpha} - \bar{\nabla}_\alpha h_{\mu\beta}) - (\bar{\nabla}_\beta h^{\alpha\beta} - \frac{1}{2}\bar{\nabla}^\alpha h^\beta_{\ \beta}) \\ &\times (\bar{\nabla}_\nu h_{\mu\alpha} + \bar{\nabla}_\mu h_{\nu\alpha} - \bar{\nabla}_\alpha h_{\mu\nu})),\end{aligned} \tag{136}$$

where all covariant derivatives are just the ones associated to the background metric, and the indices are raised and lowered using the background metric. In this expression no choice of gauge has been made yet. We thus start by imposing the same conditions as we did before, given by (117). In equation (136), however, we still have just the covariant derivatives, and not the partials as in the condition $\partial^i h_{ij} = 0$ from (117). It is therefore more convenient to first show that, in the case of a FLRW background metric, the three conditions from (117) together imply the condition $\nabla^\alpha h_{\alpha\beta} = 0$, and plug this into (136). We start by computing the contraction of the perturbations with a covariant derivative:

$$\nabla^\alpha h_{\alpha\beta} = \partial^\alpha h_{\alpha\beta} - g^{\alpha\lambda}\Gamma^\kappa_{\alpha\lambda}h_{\kappa\beta} - g^{\alpha\lambda}\Gamma^\kappa_{\beta\lambda}h_{\kappa\alpha}. \tag{137}$$

Now we can start eliminating terms: $\partial^i h_{ij} = 0$ and $h_{0\nu} = 0$ together imply $\partial^\alpha h_{\alpha\beta} = 0$ and looking at (94) we see that because the Christoffel symbols in the second term are forced by the diagonality of the metric in front to have identical lower indices they will only be non-zero for $\kappa = 0$, but then $h_{0\beta} = 0$ sets this term to zero. This leaves us with, splitting the sums in 0 components and latin indices:

$$\nabla^\alpha h_{\alpha\beta} = -g^{0\lambda}\Gamma^\kappa_{\beta\lambda}h_{0\lambda} - g^{i0}\Gamma^\kappa_{\beta 0}h_{\kappa i} - g^{ij}\Gamma^0_{\beta j}h_{0i} - g^{ij}\Gamma^k_{\beta j}h_{ki}, \tag{138}$$

where now the first and third terms vanish by the transverse condition on $h_{\mu\nu}$, the second by diagonality of $g_{\mu\nu}$, and the last term yields, using (94) for the Christoffel symbols:

$$\nabla^\alpha h_{\alpha\beta} = -a\dot{a}g^{ij}h_{ij} = -a\dot{a}h^i_{\ i} = 0. \tag{139}$$

Plugging this in to (136), together with the conditions from (117):

$$\begin{aligned}\delta^2 R^{(TT)}_{\mu\nu} &= \frac{1}{2}(\frac{1}{2}\bar{\nabla}_\mu h_{\alpha\beta}\bar{\nabla}_\nu h^{\alpha\beta} + h^{\alpha\beta}(\bar{\nabla}_\mu\bar{\nabla}_\nu h_{\alpha\beta} + \bar{\nabla}_\alpha\bar{\nabla}_\beta h_{\mu\nu} - \bar{\nabla}_\nu\bar{\nabla}_\beta h_{\alpha\mu} \\ &- \bar{\nabla}_\mu\bar{\nabla}_\beta h_{\nu\alpha}) + \bar{\nabla}^\beta h_\nu^{\ \alpha}(\bar{\nabla}_\beta h_{\mu\alpha} - \bar{\nabla}_\alpha h_{\mu\beta})),\end{aligned} \tag{140}$$

Taking the trace to find the Ricci-scalar:

$$\begin{aligned}
\delta^2 R^{(TT)} &= g^{\mu\nu}\delta^2 R^{(TT)}_{\mu\nu} - h^{\mu\nu}\delta R^{(TT)}_{\mu\nu} \\
&= \frac{1}{2}(\frac{1}{2}\bar{\nabla}^\mu h_{\alpha\beta}\bar{\nabla}_\mu h^{\alpha\beta} + h^{\alpha\beta}(\bar{\nabla}^\mu\bar{\nabla}_\mu h_{\alpha\beta} - 2\bar{\nabla}^\mu\bar{\nabla}_\beta h_{\alpha\mu}) \\
&+ \bar{\nabla}^\beta h^{\mu\alpha}\bar{\nabla}_\beta h_{\mu\alpha} - \bar{\nabla}^\beta h^{\mu\alpha}\bar{\nabla}_\alpha h_{\mu\beta}) \\
&= \frac{1}{2}(\frac{3}{2}\bar{\nabla}^\mu h_{\alpha\beta}\bar{\nabla}_\mu h^{\alpha\beta} + h^{\alpha\beta}(\bar{\nabla}^\mu\bar{\nabla}_\mu h_{\alpha\beta} - 2\bar{\nabla}^\mu\bar{\nabla}_\beta h_{\alpha\mu}) \\
&- \bar{\nabla}^\beta h^{\mu\alpha}\bar{\nabla}_\alpha h_{\mu\beta}). \qquad (141)
\end{aligned}$$

Now we proceed with taking the average. It can be shown [14] that for an averaging the following holds up to the order we are considering:

$$\langle h_{\alpha\beta}\nabla_\mu\nabla_\nu h_{\rho\sigma}\rangle = \langle h_{\alpha\beta}\nabla_\nu\nabla_\mu h_{\rho\sigma}\rangle = -\langle\nabla_\mu h_{\alpha\beta}\nabla_\nu h_{\rho\sigma}\rangle. \qquad (142)$$

Using this, we get for the average of the Ricci-tensor, plugging in gauge conditions wherever we can:

$$\begin{aligned}
\langle\delta^2 R^{(TT)}_{\mu\nu}\rangle &= \frac{1}{2}\langle\frac{1}{2}\bar{\nabla}_\mu h_{\alpha\beta}\bar{\nabla}_\nu h^{\alpha\beta} - \bar{\nabla}_\mu h^{\alpha\beta}\bar{\nabla}_\nu h_{\alpha\beta} - \bar{\nabla}_\alpha h^{\alpha\beta}\bar{\nabla}_\beta h_{\mu\nu} \\
&+ \bar{\nabla}_\beta h^{\alpha\beta}\bar{\nabla}_\nu h_{\alpha\mu} + \bar{\nabla}_\beta h^{\alpha\beta}\bar{\nabla}_\mu h_{\nu\alpha} + (\bar{\nabla}^\beta\bar{\nabla}_\beta h_\nu^{\ \alpha})h_{\mu\alpha} \\
&+ (\bar{\nabla}^\beta\bar{\nabla}_\alpha h_\nu^{\ \alpha})h_{\mu\beta}\rangle \\
&= \frac{1}{2}\langle-\frac{1}{2}\bar{\nabla}_\mu h_{\alpha\beta}\bar{\nabla}_\nu h^{\alpha\beta} + (\bar{\nabla}^\beta\bar{\nabla}_\beta h_\nu^{\ \alpha})h_{\mu\alpha}\rangle, \qquad (143)
\end{aligned}$$

and for the average of the Ricci-scalar:

$$\begin{aligned}
\langle\delta^2 R^{(TT)}\rangle &= \frac{1}{2}\langle-\frac{3}{2}h_{\alpha\beta}\bar{\nabla}^\mu\bar{\nabla}_\mu h^{\alpha\beta} + h^{\alpha\beta}\bar{\nabla}^\mu\bar{\nabla}_\mu h_{\alpha\beta} - 2h^{\alpha\beta}\bar{\nabla}_\beta\bar{\nabla}^\mu h_{\alpha\mu} \\
&- \bar{\nabla}^\beta\bar{\nabla}_\alpha h^{\mu\alpha}h_{\mu\beta}\rangle \\
&= -\frac{1}{4}\langle h_{\alpha\beta}\bar{\nabla}^\mu\bar{\nabla}_\mu h^{\alpha\beta}\rangle. \qquad (144)
\end{aligned}$$

In order to give full insight into the calculations needed to find the equations of motion (132), we took the low-brow route of computing directly, also because we were not working in a vacuum. In [14] the vacuum case is considered, which gives for the propagation equation in the transverse traceless gauge:

$$\nabla^\mu\nabla_\mu h_{ij} = 0. \qquad (145)$$

Following [14] one would then insert this into equations (143) and (144) to give a simpler form for $T^{(\mathrm{GW})}_{\mu\nu}$. The equation we have for freely propagating waves is however $\Box\tilde{h}_{ij} = 0$. The error made in using this to assume (145), however, is of the order of magnitude of the unperturbed Riemann-tensor multiplied by h_{ij}, which, inside the average, can be ignored by the same arguments used to justify (142). Proceeding with plugging (145) into (144) sets $\langle\delta^2 R^{(TT)}\rangle = 0$, and we thus get for $\langle\delta^2 G_{\mu\nu}\rangle$, in the case of freely propagating waves:

$$\langle\delta^2 G_{\mu\nu}\rangle = \langle\delta^2 R^{(TT)}_{\mu\nu}\rangle = -\frac{1}{4}\langle\bar{\nabla}_\mu h_{\alpha\beta}\bar{\nabla}_\nu h^{\alpha\beta}\rangle. \qquad (146)$$

We thus get from (135) for the energy-momentum tensor of freely propagating gravitational waves in the transverse-traceless gauge:

$$T^{(\mathrm{GW})}_{\mu\nu} = \frac{-1}{32\pi G}\left\langle\bar{\nabla}_\mu h_{\alpha\beta}\bar{\nabla}_\nu h^{\alpha\beta}\right\rangle. \qquad (147)$$

5.3 Energy Density in FLRW-Universe

With equation (147) in hand, we can give an expression for the energy density due to gravitational waves, even though it will cost us doing a little algebra. Recall that the energy density is the 00-component in the energy-momentum tensor, so:

$$\begin{aligned}
\rho_{\rm gw} &= \frac{-1}{32\pi G}\langle\bar{\nabla}_0 h_{\alpha\beta}\bar{\nabla}_0 h^{\alpha\beta}\rangle \\
&= \frac{-1}{32\pi G}\langle(\dot{h}_{\alpha\beta}-\bar{\Gamma}^{\lambda}_{0\alpha}h_{\lambda\beta}-\bar{\Gamma}^{\lambda}_{0\beta}h_{\lambda\alpha})(\dot{h}^{\alpha\beta}+\bar{\Gamma}^{\alpha}_{0\kappa}h^{\kappa\beta}+\bar{\Gamma}^{\beta}_{0\kappa}h^{\kappa\alpha})\rangle \\
&= \frac{-1}{16\pi G}\langle\frac{1}{2}\dot{h}_{\alpha\beta}\dot{h}^{\alpha\beta}+\dot{h}_{\alpha\beta}\bar{\Gamma}^{\alpha}_{0\kappa}h^{\kappa\beta}-\bar{\Gamma}^{\lambda}_{0\alpha}h_{\lambda\beta}\dot{h}^{\alpha\beta}-\bar{\Gamma}^{\lambda}_{0\alpha}h_{\lambda\beta}\bar{\Gamma}^{\alpha}_{0\kappa}h^{\kappa\beta} \\
&- \bar{\Gamma}^{\lambda}_{0\alpha}h_{\lambda\beta}\bar{\Gamma}^{\beta}_{0\kappa}h^{\kappa\alpha}\rangle. \qquad (148)
\end{aligned}$$

We can clean this up by realizing that the only non-zero components of the Christoffel symbols are those with two latin indices and one index zero, and $h_{0i}=0$, and that $h^{\mu\nu}=-\bar{g}^{\mu\beta}\bar{g}^{\nu\alpha}h_{\beta\alpha}$, thus that $h^{ij}=-\frac{1}{a^4}h_{ij}$, and $\dot{h}^{ij}=\frac{4\dot{a}}{a^5}h_{ij}-\frac{1}{a^4}\dot{h}_{ij}$. This cancels the $\dot{h}_{ij}$ parts of the second and the third term:

$$-\frac{1}{a^4}\dot{h}_{ab}\Gamma^{a}_{0k}h_{bk}+\frac{1}{a^4}\Gamma^{l}_{0a}h_{lb}\dot{h}_{ab}=0, \qquad (149)$$

leaving us with:

$$\rho_{\rm gw}=\frac{1}{16\pi G}\langle\frac{1}{2}\dot{h}_{ab}\dot{h}^{ab}-\frac{4\dot{a}}{a^5}\bar{\Gamma}^{j}_{0a}h_{jb}h_{ab}-\bar{\Gamma}^{j}_{0a}h_{jb}\bar{\Gamma}^{a}_{0k}h^{kb}-\bar{\Gamma}^{j}_{0a}h_{jb}\bar{\Gamma}^{b}_{0k}h^{ka}\rangle. \quad (150)$$

Plugging in the explicit expression for the Christoffel symbols (94) and immediately performing the summation to get rid of the Kronecker delta's we get:

$$\begin{aligned}
\rho_{\rm gw} &= \frac{1}{16\pi G}\langle\frac{1}{2a^4}\dot{h}_{ab}\dot{h}_{ab}-\frac{2\dot{a}}{a^5}\dot{h}_{ab}h_{ab}+\frac{4\dot{a}^2}{a^6}h_{ab}h_{ab}-\frac{\dot{a}^2}{a^6}h_{ab}h_{ab}-\frac{\dot{a}^2}{a^6}h_{ab}h_{ab}\rangle \\
&= \frac{1}{16\pi G}\langle\frac{1}{2a^4}\dot{h}_{ab}\dot{h}_{ab}-\frac{2\dot{a}}{a^5}\dot{h}_{ab}h_{ab}+\frac{2\dot{a}^2}{a^6}h_{ab}h_{ab}\rangle. \qquad (151)
\end{aligned}$$

Better, but still not very nice. If we now define $\tilde{h}_{ab}$ by $h_{ab}=a^2\tilde{h}_{ab}$ (so $\dot{h}_{ab}=2\dot{a}a\tilde{h}_{ab}+a^2\dot{\tilde{h}}_{ab}$), like in section 4.2, we can simplify this some more:

$$\begin{aligned}
\rho_{\rm gw} &= \frac{1}{16\pi G}\langle\frac{1}{2a^4}(4\dot{a}^2a^2\tilde{h}_{ab}\tilde{h}_{ab}+4\dot{a}a^3\dot{\tilde{h}}_{ab}\tilde{h}_{ab}+a^4\dot{\tilde{h}}_{ab}\dot{\tilde{h}}_{ab})-\frac{4\dot{a}^2}{a^2}\tilde{h}_{ab}\tilde{h}_{ab} \\
&- \frac{2\dot{a}}{a}\dot{\tilde{h}}_{ab}\tilde{h}_{ab}+\frac{2\dot{a}^2}{a^2}\tilde{h}_{ab}\tilde{h}_{ab}\rangle \\
&= \frac{1}{32\pi G}\langle\dot{\tilde{h}}_{ab}\dot{\tilde{h}}_{ab}\rangle. \qquad (152)
\end{aligned}$$

This is a rather satisfying result, it expresses the energy density of a gravitational wave in terms of an average over its time derivatives, which is a doable calculation.[1]

[1]Every article I have seen so far just quotes result (152), sometimes even with reference to [14], without mentioning any simplifications made. I find this a bit strange, and have tried to explain here as best I can how the result here is related to formula (35.70′) in [14].

5.4 Energy-Density Spectrum

Now that we have found an expression for ρ_{gw}, we can calculate the energy density spectrum for the gravitational waves. Because we have been redefining how we represent the perturbations, our notation got a bit messy. So, as a reminder, we define $\tilde{h}_{\mu\nu}$ by:

$$g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)(\delta_{ij} + \tilde{h}_{ij})dx^i dx^j, \tag{153}$$

with $\tilde{h}_{0\nu} = \partial_i \tilde{h}_{ij} = \tilde{h}_{ii} = 0$.
In order to be able to evaluate (152), we need to specify what we mean by averaging over a few wavelengths[2]. In agreement with [6] we take the average to be:

$$\langle f(\mathbf{x})\rangle_V = \frac{1}{V}\int_V d^3\mathbf{x} f(\mathbf{x}), \tag{154}$$

where, in the case of the perturbations, we take the volume V we are integrating over to be much larger than the wavelengths.
With these definitions, we start by plugging in the spatial Fourier transform for the perturbations,

$$\tilde{h}_{ij}(t,\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\tilde{h}_{ij}(t,\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}, \tag{155}$$

and the definition for the average (154) into equation (152):

$$\begin{aligned}
\rho_{\mathrm{gw}} &= \frac{1}{32\pi GV}\int_V d^3\mathbf{x}\int\frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\int\frac{d^3\mathbf{k}'}{(2\pi)^{\frac{3}{2}}}\dot{\tilde{h}}_{ij}(t,\mathbf{k})\dot{\tilde{h}}_{ij}(t,\mathbf{k}')e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} \\
&= \frac{1}{32\pi GV}\int d^3\mathbf{k}\int d^3\mathbf{k}'\dot{\tilde{h}}_{ij}(t,\mathbf{k})\dot{\tilde{h}}_{ij}(t,\mathbf{k}')\delta^{(3)}(\mathbf{k}+\mathbf{k}') \\
&= \frac{1}{32\pi GV}\int d^3\mathbf{k}\dot{\tilde{h}}_{ij}(t,\mathbf{k})\dot{\tilde{h}}_{ij}(t,-\mathbf{k}) \\
&= \frac{1}{32\pi GV}\int d^3\mathbf{k}\dot{\tilde{h}}_{ij}(t,\mathbf{k})\dot{\tilde{h}}^*_{ij}(t,\mathbf{k})
\end{aligned} \tag{156}$$

Since the range of frequencies gravitational waves that can plausibly be detected ([9]) is so large, it is convenient to express the energy-density spectrum in terms of the logarithmic frequency interval: $(\frac{d\rho_{\mathrm{gw}}}{d(\ln k)})$. To get there, it would be nice if we could express ρ_{gw} in terms of an integral over $\ln(k)$. It turns out we can. First, we switch to spherical coordinates (k,θ,ϕ):

$$\int_\infty^\infty d^3\mathbf{k} = \int_0^\infty dk k^2\int_0^\pi d\theta\int_0^{2\pi}d\phi\sin\theta =: \int dk k^2 d\Omega. \tag{157}$$

Observing we have $d(\ln k) = \frac{dk}{k}$, and plugging in to (156):

$$\rho_{\mathrm{gw}} = \frac{1}{32\pi GV}\int d(\ln k)d\Omega k^3\dot{\tilde{h}}_{ij}(t,\mathbf{k})\dot{\tilde{h}}^*_{ij}(t,\mathbf{k}), \tag{158}$$

Differentiating with respect to the logarithm of k is now an easy matter, leaving us with:

$$\frac{d\rho_{\mathrm{gw}}}{d(\ln k)} = \frac{k^3}{32\pi GV}\int d\Omega\dot{\tilde{h}}_{ij}(t,\mathbf{k})\dot{\tilde{h}}^*_{ij}(t,\mathbf{k}). \tag{159}$$

[2]In [14] one finds much more elaborate schemes for taking the average of a tensor field than we will use here. No justification is given for the averaging schemes we will consider here in the articles I found them in ([6] and [4]).

5.5 Spectrum for Creation during Radiation Era

The final goal of this thesis is to apply the theory developed so far to the case of bubble collisions during phase transitions that occurred after inflation, that is, during the radiation era. We are interested in how the energy-density spectrum depends on the energy-momentum tensor associated with the bubble-collisions. In order to calculate this, we need to solve the equation of motion (132) for the period the source is active, and the propagation equation ((132) with $\Pi_{ij}^{(TT)} = 0$) for after the source stopped emitting gravitational waves, and stitch the two solutions together, and put what we find into (159), and all of this during radiation era. Solving in under the assumption the universe is radiation dominated is rather convenient, for the equation of motion simplifies quite a lot, and with the the associated Green's function.
In the notation from the previous paragraph equation (132) becomes:

$$\begin{aligned} 16\pi G a^2 \Pi_{ij}^{(\mathrm{TT})} &= -a^2 \Box \tilde{h}_{ij} \\ &= -\partial_i \partial_i \tilde{h}_{ij} + a^2 \ddot{\tilde{h}}_{ij} + 3a\dot{a}\dot{\tilde{h}}_{ij}. \end{aligned} \tag{160}$$

In order not to have to carry around a^2 all the time, we switch back to the transverse traceless part of the energy-momentum tensor instead of $\Pi_{ij}^{(\mathrm{TT})}$:

$$T_{ij}^{\mathrm{TT}} = a^2 \Pi_{ij}^{(\mathrm{TT})} \tag{161}$$

Fourier transforming (155) gives us, with the definition $k^2 = \mathbf{k}^2$ for the comoving wave number k:

$$16\pi G T_{ij}^{(\mathrm{TT})}(\mathbf{k}) = k^2 \tilde{h}_{ij}(\mathbf{k}) + a^2 \ddot{\tilde{h}}_{ij}(\mathbf{k}) + 3a\dot{a}\dot{\tilde{h}}_{ij}(\mathbf{k}). \tag{162}$$

In terms of conformal time this becomes:

$$16\pi G T_{ij}^{(\mathrm{TT})}(\mathbf{k}) = k^2 \tilde{h}_{ij}(\mathbf{k}) + \tilde{h}''_{ij}(\mathbf{k}) + 2\frac{a'}{a}\tilde{h}'_{ij}(\mathbf{k}). \tag{163}$$

With the definition $\hat{h}_{ij} = a\tilde{h}_{ij}$ (thus $\tilde{h}'_{ij} = \frac{\hat{h}'_{ij}}{a} - \frac{\hat{h}_{ij}a'}{a^2}$ and $\tilde{h}''_{ij} = \frac{\hat{h}''_{ij}}{a} - \frac{2\hat{h}'_{ij}a'}{a^2} - \frac{\hat{h}_{ij}a''}{a^2} + \frac{2\hat{h}_{ij}a'^2}{a^3}$) we get:

$$\begin{aligned} 16\pi G T_{ij}^{(\mathrm{TT})}(\mathbf{k}) &= \frac{k^2\hat{h}_{ij}(\mathbf{k})}{a} + \frac{\hat{h}''_{ij}(\mathbf{k})}{a} - \frac{2\hat{h}'_{ij}(\mathbf{k})a'}{a^2} - \frac{\hat{h}_{ij}(\mathbf{k})a''}{a^2} \\ &\quad + \frac{2\hat{h}_{ij}(\mathbf{k})a'^2}{a^3} + \frac{2\hat{h}'_{ij}(\mathbf{k})a'}{a^2} - \frac{2\hat{h}_{ij}(\mathbf{k})a'^2}{a^3} \\ &= \frac{k^2\hat{h}_{ij}(\mathbf{k})}{a} + \frac{\hat{h}''_{ij}(\mathbf{k})}{a} - \frac{\hat{h}_{ij}(\mathbf{k})a''}{a^2} \\ 16\pi G a T_{ij}^{(\mathrm{TT})}(\mathbf{k}) &= (k^2 - \frac{a''}{a})\hat{h}_{ij}(\mathbf{k}) + \hat{h}''_{ij}(\mathbf{k}). \end{aligned} \tag{164}$$

If we now assume radiation domination[3] ($\frac{\dot{a}}{a} = \frac{1}{2t}$) we get for $\frac{a''}{a}$:

$$\left(\frac{\dot{a}}{a}\right)^{\cdot} = \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} = \frac{-1}{2t^2}, \tag{165}$$

[3] In [6], radiation domination is not assumed, because there the case of preheating after inflation is considered, and alternative arguments are presented to neglect the a''/a term. For the treatment of gravitational radiation from bubble collisions during phase transitions, however, radiation domination is a valid assumption for most early-universe phase transitions.

so:

$$\frac{\ddot{a}}{a} = \frac{-1}{2t^2} + \frac{1}{4t^2} = \frac{-1}{4t^2}, \tag{166}$$

and:

$$\begin{aligned} \frac{a''}{a^3} &= \frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} \\ &= \frac{1}{4t^2} - \frac{1}{4t^2} \\ &= 0, \end{aligned} \tag{167}$$

so, since a is finite, we get for the equation of motion (164):

$$16\pi G a T_{ij}^{(\mathrm{TT})}(\mathbf{k}) = k^2 \hat{h}_{ij}(\mathbf{k}) + \hat{h}''_{ij}(\mathbf{k}), \tag{168}$$

which is just a simple wave equation that can be solved by using a Green's function. Under the assumption that before the time the source started radiating at time τ_i there were no gravitational waves present ($h_{ij}(\tau_i) = h'_{ij}(\tau_i) = 0$), this Green's function for a given $k \neq 0$ is given by:

$$G(\tau - \tau') = \frac{1}{k} \sin(k(\tau - \tau')), \tag{169}$$

So we can construct the solution:

$$\hat{h}_{ij}(\tau, \mathbf{k}) = \frac{16\pi G}{k} \int_{\tau_i}^{\tau} d\tau' \sin(k(\tau - \tau')) a(\tau') T_{ij}^{(\mathrm{TT})}(\tau', \mathbf{k}). \tag{170}$$

At some time τ_f the source will stop radiating, or at least the radiation will become negligible. Then we need to solve equation (168) without the source, and match the solution with $h(\tau_f, \mathbf{k})$ from equation (170). The solutions for (168) without a source (so $\tau \geq \tau_f$) are simply plane waves:

$$\hat{h}_{ij}(\tau, \mathbf{k}) = A_{ij}(\mathbf{k}) \sin(k(\tau - \tau')) + B_{ij}(\mathbf{k}) \cos(k(\tau - \tau')). \tag{171}$$

We can determine $A_{ij}(\mathbf{k})$ and $B_{ij}(\mathbf{k})$ by demanding the solution for $\tau \leq \tau_f$ matches the solution for $\tau \geq \tau_f$, that is:

$$B_{ij}(\mathbf{k}) = \frac{16\pi G}{k} \int_{\tau_i}^{\tau_f} d\tau' \sin(k(\tau_f - \tau')) a(\tau') T_{ij}^{(\mathrm{TT})}(\tau', \mathbf{k}). \tag{172}$$

And demanding that the derivatives also match:

$$\begin{aligned} A_{ij}(\mathbf{k}) &= \frac{1}{k} \frac{d}{d\tau} |_{\tau=\tau_f} \frac{16\pi G}{k} \int_{\tau_i}^{\tau} d\tau' \sin(k(\tau - \tau')) a(\tau') T_{ij}^{(\mathrm{TT})}(\tau', \mathbf{k}) \\ &= \frac{16\pi G}{k^2} \left(0 + k \int_{\tau_i}^{\tau_f} d\tau' \cos(k(\tau_f - \tau')) a(\tau') T_{ij}^{(\mathrm{TT})}(\tau', \mathbf{k}) \right) \\ &= \frac{16\pi G}{k} \int_{\tau_i}^{\tau_f} d\tau' \cos(k(\tau_f - \tau')) a(\tau') T_{ij}^{(\mathrm{TT})}(\tau', \mathbf{k}). \end{aligned} \tag{173}$$

To compute the energy-density spectrum, we want to plug this into equation (159). This equation, however, is in terms of cosmic time and h_{ij}. Fortunately, if we look at (152) where we started from,

$$\rho_{\mathrm{gw}} = \frac{1}{32\pi G} \langle \dot{h}_{ab} \dot{h}_{ab} \rangle, \tag{174}$$

and rewrite this in terms of conformal time and $\hat{h}_{ij}$, using $\dot{h}_{ij} = -\frac{a'}{a^3}\hat{h}_{ij} + \frac{1}{a^2}\hat{h}'_{ij}$, we get:

$$\rho_{\rm gw} = \frac{1}{32\pi G a^4}\langle \frac{a'^2}{a^2}\hat{h}_{ij}\hat{h}_{ij} - \frac{2a'}{a}\hat{h}_{ij}\hat{h}'_{ij} + \hat{h}'_{ij}\hat{h}'_{ij}\rangle. \tag{175}$$

In terms of the Hubble-parameter ($H = \frac{a'}{a^2}$):

$$\rho_{\rm gw} = \frac{1}{32\pi G a^4}\langle a^2 H^2 \hat{h}_{ij}\hat{h}_{ij} - 2aH\hat{h}_{ij}\hat{h}'_{ij} + \hat{h}'_{ij}\hat{h}'_{ij}\rangle. \tag{176}$$

We are interested in the smaller wavelengths (much smaller than the Hubble radius), with $\frac{k}{a} \gg H$, and in this approximation the first and the second term are much smaller than the last. We thus approximate:

$$\rho_{\rm gw} = \frac{1}{32\pi G a^4}\langle \hat{h}'_{ij}\hat{h}'_{ij}\rangle + \mathcal{O}(k/(aH)), \tag{177}$$

and we see that the previous results carry over, with as only difference a factor of $\frac{1}{a^4}$. Because all radiation energy-densities dilute as $\frac{1}{a^4}$ (three powers of $\frac{1}{a}$ for spatial dilution, and one for the redshift), and we can treat gravitational waves as radiation, it makes sense to define the quantity S_k by

$$\left(\frac{d\rho_{\rm gw}}{d(\ln k)}\right) = \frac{S_k}{a^4}. \tag{178}$$

We thus get:

$$S_k = \frac{k^3}{32\pi G V}\int d\Omega \hat{h}'_{ij}(t,\mathbf{k})\hat{h}^{*\prime}_{ij}(t,\mathbf{k}). \tag{179}$$

Our next step is to determine $\hat{h}'_{ij}(t,\mathbf{k})\hat{h}^{*\prime}_{ij}(t,\mathbf{k})$. We make an another simplifying assumption: we are not interested in the energy-density spectrum at the level of the oscillations, that is, we average over one period of oscillation $T = \frac{2\pi}{k}$:

$$\langle \hat{h}'_{ij}(t,\mathbf{k})\hat{h}^{*\prime}_{ij}(t,\mathbf{k})\rangle_T = \frac{k}{2\pi}\int_0^{\frac{2\pi}{k}} dt \hat{h}'_{ij}(t,\mathbf{k})\hat{h}^{*\prime}_{ij}(t,\mathbf{k}). \tag{180}$$

We have:

$$\hat{h}'_{ij}(\tau,\mathbf{k}) = kA_{ij}(\mathbf{k})\cos(k(\tau-\tau_f)) - kB_{ij}(\mathbf{k})\sin(k(\tau-\tau_f)), \tag{181}$$

so:

$$\begin{aligned}
\hat{h}'_{ij}(\tau,\mathbf{k})\hat{h}^{*\prime}_{ij}(\tau,\mathbf{k}) &= k^2\sum_{i,j}(|A_{ij}(\mathbf{k})|^2\cos^2(k(\tau-\tau_f)) \\
&+ |B_{ij}(\mathbf{k})|^2\sin^2(k(\tau-\tau_f)) \\
&+ (A^*_{ij}(\mathbf{k})B_{ij}(\mathbf{k}) + A_{ij}(\mathbf{k})B^*_{ij}(\mathbf{k})) \\
&\times \cos(k(\tau-\tau_f))\sin(k(\tau-\tau_f))).
\end{aligned} \tag{182}$$

Computing the integral gives zero for the cross-terms, and the squares of both the sine and the cosine integrate to $\frac{\pi}{k}$. This leaves us with:

$$\begin{aligned}
\langle \hat{h}'_{ij}(\tau,\mathbf{k})\hat{h}^{*\prime}_{ij}(\tau,\mathbf{k})\rangle_T &= \frac{k^2}{2}\sum_{i,j}\left(|A_{ij}(\mathbf{k})|^2+|B_{ij}(\mathbf{k})|^2\right)\\
&= \frac{(16\pi G)^2}{2}\\
&\times \sum_{ij}(|\int_{\tau_i}^{\tau_f} d\tau' \cos(k(\tau_f-\tau'))a(\tau')T^{(\mathrm{TT})}_{ij}(\tau',\mathbf{k})|^2\\
&+ |\int_{\tau_i}^{\tau_f} d\tau' \sin(k(\tau_f-\tau'))a(\tau')T^{(\mathrm{TT})}_{ij}(\tau',\mathbf{k})|^2) \quad (183)
\end{aligned}$$

Using $\cos(k(\tau_f-\tau')) = \cos(k\tau_f)\cos(k\tau')+\sin(k\tau_f)\sin(k\tau')$ and $\sin(k(\tau_f-\tau')) = -\cos(k\tau_f)\sin(k\tau')+\sin(k\tau_f)\cos(k\tau')$, we can take the $\cos(k\tau_f)$ and $\sin(k\tau_f)$ factors out of the integrals, which gives two times $\cos^2(k\tau_f)+\sin^2(k\tau_f)=1$, one for $\sin(k\tau')$ and one for $\cos(k\tau')$ in the integral. Plugging this into (179), we get for S_k:

$$\begin{aligned}
S_k &= \frac{4\pi G k^3}{V}\int d\Omega \sum_{i,j}(|\int_{\tau_i}^{\tau_f} d\tau' \cos(k\tau')a(\tau')T^{(\mathrm{TT})}_{ij}(\tau',\mathbf{k})|^2\\
&+ |\int_{\tau_i}^{\tau_f} d\tau' \sin(k\tau')a(\tau')T^{(\mathrm{TT})}_{ij}(\tau',\mathbf{k})|^2). \quad (184)
\end{aligned}$$

We have thus found an expression for the logarithmic energy-density spectrum of gravitational waves created during the radiation era in terms of the energy-momentum tensor of its source. There is an alternative way of defining the average ([6]), which is particularly convenient when dealing with stochastic sources as we will see in the next chapter. This method is called the ensemble average, which is an average over the Fourier components of $\hat{h}_{ij}$, taking into account interactions between the various components, that is, we define:

$$\langle \hat{h}'_{ij}(\tau,\mathbf{x})\hat{h}'_{ij}(\tau,\mathbf{x})\rangle = \int \frac{d\mathbf{k}}{(2\pi)^{3/2}}\int \frac{d\mathbf{q}}{(2\pi)^{3/2}}\langle \hat{h}'_{ij}(\tau,\mathbf{k})\hat{h}^{*\prime}_{ij}(\tau,\mathbf{q})\rangle e^{-i(\mathbf{k}-\mathbf{q})\cdot\mathbf{x}}. \quad (185)$$

Note that with our previous averaging scheme we got (see (156)):

$$\langle \hat{h}'_{ij}(\tau,\mathbf{x})\hat{h}'_{ij}(\tau,\mathbf{x})\rangle = \frac{1}{V}\int d\mathbf{k}\hat{h}'_{ij}(\tau,\mathbf{k})\hat{h}'^{*}_{ij}(\tau,\mathbf{k}), \quad (186)$$

and we have the following relation between the two averages (by the ergodic assumption the ensemble average should be equivalent to the space average):

$$\frac{1}{V}\int d\mathbf{k}\hat{h}'_{ij}(\tau,\mathbf{k})\hat{h}'^{*}_{ij}(\tau,\mathbf{k}) = \frac{1}{(2\pi)^3}\int d\mathbf{k}\int d\mathbf{q}\langle \hat{h}'_{ij}(\tau,\mathbf{k})\hat{h}^{*\prime}_{ij}(\tau,\mathbf{k})\rangle e^{-i(\mathbf{k}-\mathbf{q})\cdot\mathbf{x}} \quad (187)$$

Now we define the unequal time correlator for the tensor anisotropic stress in Fourier space $\Pi(k,\tau,\zeta)$ by:

$$\langle T^{(\mathrm{TT})}_{ij}(\tau,\mathbf{k})T^{*(\mathrm{TT})}_{ij}(\zeta,\mathbf{q})\rangle = \delta(\mathbf{k}-\mathbf{q})\Pi(k,\tau,\zeta), \quad (188)$$

where we note that this definition only makes sense with the assumption that the polarization of the gravitational waves is of no interest to us, thus assuming homogeneity and isotropy in Fourier space (see [4]). Combining the above, taking the tensor to be averaged over to be $T_{ij}^{(\mathrm{TT})}(\tau, \mathbf{k})$, we get:

$$\frac{1}{V} \int d\mathbf{k} T_{ij}^{(\mathrm{TT})}(\tau, \mathbf{k}) T_{ij}^{*(\mathrm{TT})}(\zeta, \mathbf{k}) = \frac{1}{(2\pi)^3} \int d\mathbf{k} \Pi(k, \tau, \zeta), \tag{189}$$

where we already performed the $\mathbf{q}$ integral to get rid of the delta function. In equation (189) we are dealing with unequal times, where in (187) we had equal times. This, however, is of no consequence: the definitions in (185) and (186) and thus the result in (187) also make sense for unequal times.
The finishing touch in the discussion of this alternative way of averaging is of course expressing S_k in terms of it. If we try to do this directly from (184) we find we would need a relation between $T_{ij}^{(\mathrm{TT})}(\tau, \mathbf{k}) T_{ij}^{*(\mathrm{TT})}(\zeta, \mathbf{k})$ and $\Pi(k, \tau, \zeta)$, where (189) is only a relation of integrals. However, looking at the calculations we did to find S_k, we see that setting

$$\frac{1}{V} T_{ij}^{(\mathrm{TT})}(\tau, \mathbf{k}) T_{ij}^{*(\mathrm{TT})}(\zeta, \mathbf{k}) = \frac{1}{(2\pi)^3} \Pi(k, \tau, \zeta) \tag{190}$$

in expression (184) for S_k just corresponds with using the identity (189) before we switch to spherical coordinates and differentiate with respect to $\ln(k)$ in the computations for S_k. Using (190) together with the familiar $\cos(k\tau')\cos(k\zeta) + \sin(k\tau')\sin(k\zeta) = \cos(k\tau' - k\zeta)$ we can rewrite (184) as:

$$\begin{aligned} S_k &= \frac{4\pi G k^3}{V} \int d\Omega \int_{\tau_i}^{\tau_f} d\tau' \int_{\tau_i}^{\tau_f} d\zeta a(\tau') a(\zeta) T_{ij}^{(\mathrm{TT})}(\tau, \mathbf{k}) T_{ij}^{*(\mathrm{TT})}(\zeta, \mathbf{k}) \\ &\times \ (\cos(k\tau')\cos(k\zeta) + \sin(k\tau')\sin(k\zeta)) \\ &= \frac{2Gk^3}{\pi} \int_{\tau_i}^{\tau_f} d\tau' \int_{\tau_i}^{\tau_f} d\zeta \cos(k\tau' - k\zeta) a(\tau') a(\zeta) \Pi(k, k\tau', k\zeta), \end{aligned} \tag{191}$$

where we performed the angular integral to get 4π.
A simplification we can make is to assume that the source is active for a time much shorter than the Hubble time. We can then neglect the expansion of the universe during the time the source is active, and take the scale factor in the above integral to be constant with value a_*:

$$S_k = \frac{2Gk^3 a_*^2}{\pi} \int_{\tau_i}^{\tau_f} d\tau' \int_{\tau_i}^{\tau_f} d\zeta \cos(k\tau' - k\zeta) \Pi(k, k\tau', k\zeta). \tag{192}$$

This is the expression for S_k we will use to compute the spectrum for bubble collisions during phase transitions.

5.6 Energy Density Spectrum Today

With equation (184) in hand, we can proceed to derive an expression for the spectrum per logarithmic frequency interval of the abundance of gravitational wave energy-density today. Because the gravitational wave energy-density dissipates as radiation we can do this by just considering the evolution of of the scale factor a. In general, the abundance of energy-density for a given species l in cosmology is given in the form:

$$h^2\Omega_{\rm l} = h^2\frac{\rho_{\rm l}}{\rho_{\rm c}}, \tag{193}$$

where h^2 is the astronomical length scale and ρ_c is the critical energy density. We now define $\Omega_{\rm gw}(f)$ to be the spectrum of the gravitational wave energy-density today:

$$h^2\frac{d\Omega_{\rm gw}(f)}{d\ln(f)} = \left(\frac{h^2}{\rho_c}\frac{d\rho_{\rm gw}}{d\ln(f)}\right)_0, \tag{194}$$

with f denoting the physical frequency today, and the subscript 0 reminding us we are evaluating the relevant quantities today. The wave-vector $\mathbf{k}$ we have in equation (184) is conjugate to the vector $\mathbf{x}$, and this a vector in the so-called comoving frame. So k^2 in (184) is the square of the comoving wave number, and we need to take out scale factor to obtain the physical quantity. The physical wave number today is then given by $k_0 = \frac{k}{a_0}$ and the frequency by $f = \frac{k}{2\pi a_0}$. We thus get for the spectrum of gravitational wave energy-density today:

$$h^2\frac{d\Omega_{\rm gw}}{d\ln(k)} = \frac{h^2}{\rho_{\rm c}}\frac{S_k}{a_0^4}. \tag{195}$$

This a neat formula, but does not tell us very much. This is because a_0 depends on our choice of reference, and is not some intrinsic measurable (or computable) quantity. The way to solve this is to compare the scale factor at the end of the gravitational wave emission to the scale factor today, and describe its evolution in terms of quantities that are measurable or can be found by some model. For the ratio between the scale factor at the end of emission $a_{\rm f}$ and the scale factor today the following is obtained:

$$\frac{1}{a_0} = \left(\frac{g_{\rm f}}{g_0}\right)^{-\frac{1}{12}}\left(\frac{\rho_{\rm rad0}}{\rho_{\rm f}}\right)^{\frac{1}{4}}\frac{1}{a_{\rm f}}, \tag{196}$$

where is $\rho_{\rm f}$ the total energy-density at the end of the emission. The g's denote effectively massless degrees of freedom, we take for the ratio $\frac{g_*}{g_0} = 100$. Lastly, $\rho_{\rm rad0}$ denotes the radiation energy-density today. Assuming as above that the graviational waves are emitted during radiation domination, we can assume $\rho_{\rm f} = \rho_{\rm rad}^{\rm f}$ and we get for (195):

$$h^2\frac{d\Omega_{\rm gw}}{d\ln(k)} = \frac{h^2}{\rho_{\rm c}a_{\rm f}^4}\left(\frac{g_{\rm f}}{g_0}\right)^{-\frac{1}{3}}\frac{\rho_{\rm rad0}}{\rho_{\rm rad}^{\rm f}}S_k(\tau_{\rm f}). \tag{197}$$

The nice thing about this expression is that all the quantities in it can be measured or modeled for. We have for instance $\Omega_{\rm rad0}h^2 = \frac{h^2\rho_{\rm rad0}}{\rho_{\rm c}} = 4.3\times10^{-5}$. We now have the general tools needed to compute the energy density spectrum of gravitational waves from bubble collisions.

6 Gravitational Waves from Bubble Collisions

6.1 Motivation

According to the Standard Model of cosmology, phase transitions occurred after the Big Bang, when the universe was reheated and thermalized after inflation and started cooling down due to the regular expansion. In this dense and hot state the fundamental forces are thought to have had more symmetries than they have now. An important example of such a phase transition is the electro-weak phase transition, which could be a first-order phase transition [15], occurring at a scale of $T \sim 100\text{GeV}$, around 10^{-8} seconds after the Big Bang. During first-order phase transitions, bubbles of the new phase are formed throughout the space, which then rapidly expand and collide. These collisions will break spherical symmetry of the expansion of the bubbles. Because the bubbles form randomly, this symmetry breaking occurs anisotropically, thus giving rise to an non-zero anisotropic stress, which generates gravitational waves.

This generation of gravitational waves is of more than purely theoretical interest, because gravitational waves are barely attenuated during their propagation through the universe, and can thus give us a view of the very early universe. In the specific case of the electro-weak symmetry breaking the characteristic frequency for the gravitational waves generated by bubble collisions during this phase transition is in the range that will be covered by LISA (10^{-4}-10^{-2} Hz). Because the gravitational waves encode information on both the strength of phase transition and the temperature at which it takes place, it is possible that LISA could provide information that can help us understand this phase transition.

The energy density spectrum for gravitational waves generated by bubble collisions can be computed using the formula (197). The hard part of this calculation is determining the specific form of the anisotropic stress tensor. This uses quite a lot of hydrodynamics, and can only be treated in little detail in this text. We will limit ourselves to discussing what the relevant parameters for the bubble collisions are, and how they enter the expression for the anisotropic stress. We will have a look at how the energy density spectrum depends on these parameters, and discuss the chances for actually detecting gravitational waves from bubble collisions in early universe phase transitions, using [4] as our main source. The reader should be aware that this particular subject is a field of ongoing research, and the method of modeling presented here is just one of many. The advantage of the model presented here is that the calculations are all analytic, giving a lot insight into the process that underlies the generation of gravitational waves by bubble collisions, even if this specific model turns out to be not entirely accurate.

6.2 Velocity Dependence

Before we look at a model for the bubble-collisions, we take a look at which parameters we would want to extract from such a model. To find the energy density spectrum, we need to know the energy-momentum tensor. We assume it to have perfect fluid form:

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu} \tag{198}$$

In the end we will only be interested in the anisotropic stress part, so it suffices to look at the spatial, off-diagonal part of the energy-momentum tensor. That is, we look only at the spatial components of the energy-momentum tensor (suppressing the $g_{\mu\nu}$ factor since it is diagonal anyway):

$$\begin{aligned} T_{ab}(\mathbf{x},\tau) &= (\rho+p)U_a(\mathbf{x},\tau)U_b(\mathbf{x},\tau) \\ &= (\rho+p)\gamma^2 v_a(\mathbf{x},\tau)v_b(\mathbf{x},\tau), \end{aligned} \tag{199}$$

where $\gamma^2 = 1/(1-v^2)$ is the gamma factor. For simplicity, the spatial dependence of $\rho(\tau)+p(\tau)=w(\tau)$ and γ is ignored and Fourier transforming we get:

$$\begin{aligned} \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} T_{ab}(\mathbf{k},\tau)e^{-ik\cdot x} &= \frac{w(\tau)}{1-v^2(\tau)} \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}} v_a(\mathbf{k},\tau)v_b(\mathbf{p},\tau) \\ &\times\ e^{-ix\cdot(\mathbf{k}+\mathbf{p})} \\ &= \frac{w(\tau)}{1-v^2(\tau)} \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}} v_a(\mathbf{k}-\mathbf{p},\tau)v_b(\mathbf{p},\tau) \\ &\times\ e^{-ix\cdot\mathbf{k}}. \end{aligned} \tag{200}$$

Using orthogonality and completeness of the Fourier basis yields:

$$T_{ab}(\mathbf{k},\tau) = \frac{w(\tau)}{1-v^2(\tau)} \int \frac{d^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}} v_a(\mathbf{k}-\mathbf{p},\tau)v_b(\mathbf{p},\tau). \tag{201}$$

More specifically, looking at expression (191), we see that we need to find the unequal-time correlator $\Pi(k,\tau,\zeta)$. To compute the correlator, we need to find $\langle T_{ij}^{(\mathrm{TT})}(\mathbf{k},\tau)T_{ij}^{*(\mathrm{TT})}(\mathbf{q},\zeta)\rangle$. We know from equation (117):

$$T_{ij}^{(\mathrm{TT})}(\mathbf{k},\tau) = (P_{il}P_{jm} - \frac{1}{2}P_{ij}P_{lm})T_{lm}(\mathbf{k},\tau), \tag{202}$$

where we have $P_{ij} = \delta_{ij} - \mathbf{k}_i\mathbf{k}_j$. For future convenience of notation we define:

$$P_{abcd}B_{ab}(\mathbf{k})B_{cd}(\mathbf{q}) = (P_{ia}P_{jb} - \frac{1}{2}P_{ij}P_{ab})(\mathbf{k})(P_{ic}P_{jd} - \frac{1}{2}P_{ij}P_{cd})(\mathbf{q})B_{ab}(\mathbf{k})B_{cd}(\mathbf{q}), \tag{203}$$

for $B_{ab}(\mathbf{k})$ a tensor. The expectation value of a product of transverse-traceless parts is then related to that of a product of energy-momentum tensors by:

$$\begin{aligned} \langle T_{ij}^{(\mathrm{TT})}(\mathbf{k},\tau)T_{ij}^{*(\mathrm{TT})}(\mathbf{q},\zeta)\rangle &= (P_{ia}P_{jb} - \frac{1}{2}P_{ij}P_{ab})(\mathbf{k})(P_{ic}P_{jd} - \frac{1}{2}P_{ij}P_{cd})(\mathbf{q}) \\ &\times\ \langle T_{ab}(\mathbf{k},\tau)T_{cd}^*(\mathbf{q},\zeta)\rangle \\ &= P_{abcd}(\mathbf{k},\mathbf{q})\langle T_{ab}(\mathbf{k},\tau)T_{cd}^*(\mathbf{q},\zeta)\rangle. \end{aligned} \tag{204}$$

To find the correlator we thus need to compute the expectation value of a product of energy-momentum tensors:

$$\begin{aligned} \langle T_{ab}(\mathbf{k},\tau)T_{cd}^*(\mathbf{q},\tau)\rangle &= \frac{w(\tau)w(\zeta)}{(1-v^2(\tau))(1-v^2(\zeta))} \int \frac{d^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3\mathbf{h}}{(2\pi)^{\frac{3}{2}}} \\ &\times\langle v_a(\mathbf{k}-\mathbf{p},\tau)v_b(\mathbf{p},\tau)v_c(\mathbf{q}-\mathbf{h},\zeta)v_d(\mathbf{h},\zeta)\rangle. \end{aligned} \tag{205}$$

If we approximate this by using Wick's theorem (this theorem is only valid for Gaussian distributions, and the velocity distribution will probably not be Gaussian, but we have to find a way to approximate the four-point correlators), we can reduce the expectation value of the product of four velocities to a sum over products of expectation values of the product of two velocities:

$$\begin{aligned}&\langle v_a(\mathbf{k}-\mathbf{p},\tau)v_b(\mathbf{p},\tau)v_c(\mathbf{q}-\mathbf{h},\zeta)v_d(\mathbf{h},\zeta)\rangle = \\ &\langle v_a(\mathbf{k}-\mathbf{p},\tau)v_b(\mathbf{p},\tau)\rangle\langle v_c(\mathbf{q}-\mathbf{h},\zeta)v_d(\mathbf{h},\zeta)\rangle \\ &+\langle v_a(\mathbf{k}-\mathbf{p},\tau)v_c(\mathbf{q}-\mathbf{h},\zeta)\rangle\langle v_b(\mathbf{p},\tau)v_d(\mathbf{h},\zeta)\rangle \\ &+\langle v_a(\mathbf{k}-\mathbf{p},\tau)v_d(\mathbf{h},\zeta)\rangle\langle v_b(\mathbf{p},\tau)v_c(\mathbf{q}-\mathbf{h},\zeta)\rangle. \end{aligned} \tag{206}$$

Note that the first term in this expression is just the product of the expectation values of the kinetic parts of two energy momentum tensors. To satisfy isotropy this expectation value should vanish, leaving us with:

$$\begin{aligned}\langle T_{ab}(\mathbf{k},\tau)T^*_{cd}(\mathbf{q},\tau)\rangle &= \frac{w(\tau)w(\zeta)}{(1-v^2(\tau))(1-v^2(\zeta))}\int\frac{d^3\mathbf{kp}}{(2\pi)^{\frac{3}{2}}}\int\frac{d^3\mathbf{h}}{(2\pi)^{\frac{3}{2}}} \\ &\times \left(\langle v_a(\mathbf{k}-\mathbf{p},\tau)v_c(\mathbf{q}-\mathbf{h},\zeta)\rangle\langle v_b(\mathbf{p},\tau)v_d(\mathbf{h},\zeta)\rangle\right. \\ &+ \left.\langle v_a(\mathbf{k}-\mathbf{p},\tau)v_d(\mathbf{h},\zeta)\rangle\langle v_b(\mathbf{p},\tau)v_c(\mathbf{q}-\mathbf{h},\zeta)\rangle\right). \end{aligned}\tag{207}$$

For convenience in the rest of the computations, we would like the space-time correlator to appear in the expression for the correlation of the two energy momentum tensors. Assuming statistical homogeneity, so the correlator depends only on the distance between two points $\mathbf{r}$, the correlator is given by:

$$C_{ab}(\mathbf{r},\tau,\zeta) = \langle v_a(\mathbf{x},\tau)v_b(\mathbf{x}+\mathbf{r},\zeta)\rangle. \tag{208}$$

In Fourier space we have the correlator $\hat{C}_{ab}(k,\tau,\zeta)$ defined by:

$$\langle v_a(\mathbf{k},\tau)v_b(\mathbf{q},\zeta)\rangle = \delta^{(3)}(\mathbf{k}-\mathbf{q})\hat{C}_{ab}(k,\tau,\zeta), \tag{209}$$

where the delta function is due to the statistical homogeneity and isotropy. These two correlators are related by:

$$\hat{C}_{ab}(k,\tau,\zeta) = \int\frac{d^3\mathbf{r}}{(2\pi)^{\frac{3}{2}}}C_{ab}(\mathbf{r},\tau,\zeta)e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{210}$$

Plugging (209) into equation (207) we get:

$$\begin{aligned}\langle T_{ab}(\mathbf{k},\tau)T^*_{cd}(\mathbf{q},\tau)\rangle &= \frac{w(\tau)w(\zeta)}{(1-v^2(\tau))(1-v^2(\zeta))}\int\frac{d^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}}\int\frac{d^3\mathbf{h}}{(2\pi)^{\frac{3}{2}}} \\ &\times(\delta^{(3)}(\mathbf{k}-\mathbf{p}-\mathbf{q}+\mathbf{h})\hat{C}_{ac}(|\mathbf{k}-\mathbf{p}|,\tau,\zeta) \\ &\times\delta^{(3)}(\mathbf{p}-\mathbf{h})\hat{C}_{bd}(p,\tau,\zeta) \\ &+\delta^{(3)}(\mathbf{k}-\mathbf{p}-\mathbf{h})\hat{C}_{ad}(|\mathbf{k}-\mathbf{p}|,\tau,\zeta) \\ &\times\delta^{(3)}(\mathbf{p}-\mathbf{q}+\mathbf{h})\hat{C}_{bc}(p,\tau,\zeta)). \end{aligned}\tag{211}$$

Performing the $\mathbf{h}$ integral, eliminating some delta functions:

$$\begin{aligned}\langle T_{ab}(\mathbf{k},\tau)T^*_{cd}(\mathbf{q},\tau)\rangle &= \frac{w(\tau)w(\zeta)}{(1-v^2(\tau))(1-v^2(\zeta))}\delta^{(3)}(\mathbf{k}-\mathbf{q})\int\frac{d^3\mathbf{p}}{(2\pi)^3} \\ &\times(\hat{C}_{ac}(|\mathbf{k}-\mathbf{p}|,\tau,\zeta)\hat{C}_{bd}(p,\tau,\zeta) \\ &+\hat{C}_{ad}(|\mathbf{k}-\mathbf{p}|,\tau,\zeta)\hat{C}_{bc}(p,\tau,\zeta)). \end{aligned}\tag{212}$$

The remaining task is to determine the velocity profiles for the bubbles in the phase transition. We will not discuss possible models for the velocity profile in to much detail, but present the relatively simple model found in [4].

6.3 Velocity Correlators

To find the velocity correlators, we will construct a model for the fluid velocity in a bubble during the phase-transition, following [4]. Here, however, we will not go into the details of the hydrodynamics, and when talking about specific velocity distributions we will always assume we are dealing with Jouget detonations. In [4] the case of deflagrations is also treated, but we will not discuss them here. The picture one should keep in mind when talking about bubbles is that of two concentric, spherical, expanding shells, with inside the inner shell the new phase with zero velocity, and between the two shells a non-zero velocity front where the phase expands. The relevant parameters in this picture are the radii of the shells and the velocities at which they expand. Because we are dealing with relativistic fluids we have to be careful with specifying in which frame we measure a certain velocity. We characterize the velocity of the outer shell, $v_{\rm out}$, by the incoming velocity of the old phase into the shell in the frame of the outer shell, and the velocity of the new phase fluid by the velocity at which the new phase leaves the outer shell, in the frame of the outer shell, $v_{\rm in}$. In the case of Jouget detonations the former velocity corresponds to the velocity at which the bubble wall expands in the rest frame of the bubble center, $v_{\rm b} = v_{\rm out}$, and the latter is equal to the sound speed, $v_{\rm in} = c_{\rm s}$, which is equal to $1/\sqrt{3}$ for relativistic fluids. Using time as parameter we can express both shell radii in terms of the respective shell velocities. We have for the radius of the outer shell $R = v_{\rm out}t$ and for the inner shell radius $r_{\rm int} = v_{\rm int}t$. The relevant parameters are shown in figure 4. For Jouget detonations we have $v_{\rm int} = c_{\rm s}$.

Ultimately, we will be interested in the velocity profile of the front between the

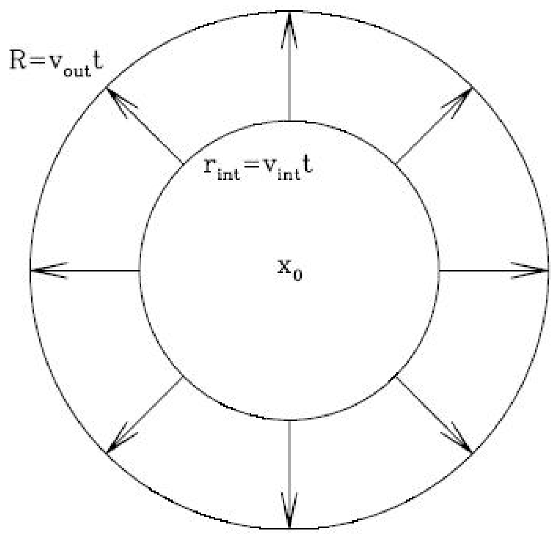

Figure 4: Velocities and radii in our model of a bubble with center $\mathbf{x}_0$. The figure is from [4].

two shells. It turns out to be convenient to consider this profile in the frame of new phase fluid. Using a Lorentz transformation we get for the fluid velocity $v_{\rm f}$ near the outer shell in the frame of the bubble center:

$$v_{\rm f} = \frac{v_{\rm out} - v_{\rm in}}{1 - v_{\rm out} v_{\rm in}}. \tag{213}$$

For simplicity we assume the velocity to increase linearly as a function of the distance to the bubble center at x_0, and normalize it to reach $v_{\rm f}$ at the outer shell. The velocity profile is then given by:

$$v_a(\mathbf{x}, t) = \left\{ \begin{array}{ll} \frac{v_{\rm f}}{R}(\mathbf{x} - \mathbf{x}_0)_a & r_{\rm int} < |\mathbf{x} - \mathbf{x}_0| < R \\ 0 & \text{otherwise} \end{array} \right\}. \tag{214}$$

With this velocity profile, we can calculate the equal time velocity correlators $\langle v_i(\mathbf{x}, t) v_j(\mathbf{y}, t)\rangle$. We do not calculate the correlators for unequal times directly, this computation is too complicated. Instead we calculate the correlators for equal times, and use these to approximate the unequal time correlators. Note that, even though here we will be focusing on the computation of the two point correlators, we are calculating an approximation for four point correlators. This is important to keep in mind, because we will take the two point correlator to be non-zero if and only if $\mathbf{x}$ and $\mathbf{y}$ are in the same bubble's non-zero velocity front, and average over all possible positions for the bubble center. At first it may seem as if this implies we will not be taking any collisions into account, collisions will certainly involve correlations between velocities in different bubbles. However, we do take these correlations into account: they appear in the four point correlator as the product of two point correlations from different bubbles. It would in fact be superfluous to try to calculate correlations between different bubbles: averaging over all different positions for the bubble centers (a space average) is by the ergodic assumption (ensemble averages are equivalent to space averages, a customary assumption in cosmology) equivalent to an average over several realizations for the center positions (an ensemble average), and with that to an average over possible bubble configurations, and in this way we also take overlapping bubbles (i.e. collisions) into account. Plugging the velocity profile (214) into the equal time correlator we get:

$$\langle v_i(\mathbf{x}, t) v_j(\mathbf{y}, t)\rangle = \frac{v_{\rm f}^2}{R(t)^2} \langle (\mathbf{x} - \mathbf{x}_0)_i (\mathbf{y} - \mathbf{x}_0)_j \rangle, \tag{215}$$

for $\mathbf{x}$ and $\mathbf{y}$ in the same bubble's non-zero velocity front, and zero otherwise. Write $V_{\rm i}$ for the volume of possible positions for x_0 such that $\mathbf{x}$ and $\mathbf{y}$ are in the same bubble's non-zero velocity front. Since the first case above gives the only non-zero contribution, we can calculate the correlator by calculating the average of the product in the right hand side of (215) over all $x_0 \in V_{\rm i}$, and then multiply by the probability of $V_{\rm i}$ actually containing a bubble center. Again using the ergodic assumption, this is the same as the fraction of volume that $V_{\rm i}$ occupies. For a point to be in $V_{\rm i}$, it should be the center of a bubble containing both $\mathbf{x}$ and $\mathbf{y}$ in its non-zero velocity front, and for that to happen it has to be the center of a bubble containing $\mathbf{x}$ and $\mathbf{y}$ in the first place. The fraction of volume filled by $V_{\rm i}$ is thus given by the fraction $V_{\rm i}$ occupies of the total volume of points such that $\mathbf{x}$ and $\mathbf{y}$ are in the same bubble $V_{\rm c}$, times the probability of a point being in a bubble which is equal to the fraction of volume occupied by

bubbles at time t, $\phi(t)$.
The correlator is then given by:

$$\langle v_i(\mathbf{x},t)v_j(\mathbf{y},t)\rangle = \phi(t)\frac{v_{\mathrm{f}}^2}{V_{\mathrm{c}}R(t)^2}\int_{V_{\mathrm{i}}} d^3\mathbf{x}_0(\mathbf{x}-\mathbf{x}_0)_i(\mathbf{y}-\mathbf{x}_0)_j, \tag{216}$$

where the V_{i}'s in the average and in the probability have canceled. The right hand side in equation (216) involves the integral over V_{i}, which depends only on the distance $r = |\mathbf{x}-\mathbf{y}|$, and the two parameters r_{int} and R. The dependence on the positions $\mathbf{x}$ and $\mathbf{y}$ separately vanishes, as we stated before, and we can thus define:

$$I_{ij}(r, r_{\mathrm{int}}, R) = \int_{V_{\mathrm{i}}} d^3\mathbf{x}_0(\mathbf{x}-\mathbf{x}_0)_i(\mathbf{y}-\mathbf{x}_0)_j, \tag{217}$$

leading to:

$$\langle v_i(\mathbf{x},t)v_j(\mathbf{y},t)\rangle = \phi(t)\frac{v_f^2}{V_{\mathrm{c}}R^2}\int_{V_{\mathrm{i}}} I_{ij}(r, r_{\mathrm{int}}, R). \tag{218}$$

To proceed from here, we have to evaluate (218) explicitly, and then Fourier transform it to obtain the correlators in Fourier space. We will not present these computations here. Instead we will have a look at convenient methods for approximating the unequal time correlators. First we discuss a naive, but physically transparent method, to get a feel for these approximations. Then we present a more complicated but more accurate method.
The first method comes down approximating the value of the unequal time correlator (208) by the value of the equal time correlator in the case that the region of non-zero velocity at comoving time τ overlaps with that at comoving time ζ, and by zero otherwise. The first step is to determine for what times the regions of non-zero velocity overlap. Suppose that $\zeta \geq \tau$. Just overlapping then means that the inner boundary of the non-zero velocity shell at ζ equals the outer boundary at τ. If we call η_{in} the time the bubble was formed, we can express this as:

$$v_{\mathrm{out}}(\tau - \eta_{\mathrm{in}}) = v_{\mathrm{int}}(\zeta - \eta_{\mathrm{in}}). \tag{219}$$

Solving this for the limiting time, and using Heaviside functions to set the correlator to zero for all ζ larger than this time, we get for the correlator with $\zeta \geq \tau$ (note that $\frac{v_{\mathrm{out}}}{v_{\mathrm{int}}} = \frac{R}{r_{\mathrm{int}}}$):

$$\langle v_i(\mathbf{x},\tau)v_j(\mathbf{y},\zeta)\rangle = \langle v_i(\mathbf{x},\tau)v_j(\mathbf{y},\tau)\rangle\Theta(\zeta-\tau)\Theta(\frac{R}{r_{\mathrm{int}}}(\tau-\eta_{\mathrm{in}})+\eta_{\mathrm{in}}-\zeta), \tag{220}$$

where we have arbitrarily set the correlator to its value at the smallest time. Symmetrizing to account for the case $\tau > \zeta$:

$$\begin{aligned}\langle v_i(\mathbf{x},\tau)v_j(\mathbf{y},\zeta)\rangle &= \langle v_i(\mathbf{x},\tau)v_j(\mathbf{y},\tau)\rangle\Theta(\zeta-\tau)\Theta(\frac{R}{r_{\mathrm{int}}}(\tau-\eta_{\mathrm{in}})+\eta_{\mathrm{in}}-\zeta)\\ &+ \langle v_i(\mathbf{x},\zeta)v_j(\mathbf{y},\zeta)\rangle\Theta(\tau-\zeta)\Theta(\frac{R}{r_{\mathrm{int}}}(\zeta-\eta_{\mathrm{in}})+\eta_{\mathrm{in}}-\tau).\end{aligned} \tag{221}$$

Even though this approximation gives a good feel for how one goes about approximating this kind of correlator, it turns out ([4]) that parts of the energy density spectrum turn out negative, where the spectrum should clearly be positive because of the $|h'_{ij}|^2$ on the left hand side of equation (183). We therefore have to look for a different way to approximate the unequal time correlators. In the calculation above we tried approximating the two point correlator for velocities. Instead of this, we can also try to approximate the unequal time correlator for the product of energy-momentum tensors $\langle T_{ab}(\mathbf{k},\tau)T^*_{cd}(\mathbf{q},\tau)\rangle$. Looking at expression (204), together with definition (188), we see that this equivalent to approximating the unequal time correlator $\Pi(k,\tau,\zeta)$. We now lack the physical context we had above to base our approximation on. Instead we will impose the condition that longer wave lengths correlate over a longer time span, which is just a manifestation of the general phenomenon that longer wave lengths attenuate slower in a medium. We further assume that the correlation does not last longer than about one wave length. To keep control of the effects of the last assumption, however, we parameterize this assumption with a positive, dimensionless parameter x_c, and assume correlation to last up to time separation x_c/k. In formula:

$$\begin{aligned}\Pi(k,\tau,\zeta) &= \Pi(k,\tau,\tau)\Theta(\zeta-\tau)\Theta(\frac{x_\mathrm{c}}{k}-(\zeta-\tau)) \\ &+ \Pi(k,\zeta,\zeta)\Theta(\tau-\zeta)\Theta(\frac{x_\mathrm{c}}{k}-(\tau-\zeta)), \end{aligned} \tag{222}$$

where $\Pi(k,\tau,\tau)$ is given by (combining equations (212) and (204) with definition (188)):

$$\begin{aligned}\Pi(k,\tau,\tau) &= \left(\frac{w(\tau)}{1-v^2(\tau)}\right)^2\int\frac{d^3\mathbf{p}}{(2\pi)^3}P_{abcd} \\ &\times (\hat{C}_{ac}(|\mathbf{k}-\mathbf{p}|,\tau,\tau)\hat{C}_{bd}(p,\tau,\tau) \\ &+ \hat{C}_{ad}(|\mathbf{k}-\mathbf{p}|,\tau,\tau)\hat{C}_{bc}(p,\tau,\tau)) \\ &= \left(\frac{w(\tau)}{1-v^2(\tau)}\right)^2\left(\frac{\phi(\tau)v_f^2}{V_\mathrm{c}R(\tau)^2}\right)^2\int\frac{d^3\mathbf{p}}{(2\pi)^6}\int d^3\mathbf{r}\int d^3\mathbf{s}P_{abcd} \\ &\times e^{i\mathbf{r}\cdot(\mathbf{k}-\mathbf{p})}e^{i\mathbf{s}\cdot\mathbf{p}}(I_{ac}(r,r_\mathrm{int},R)I_{bd}(s,r_\mathrm{int},R) \\ &+ I_{ad}(r,r_\mathrm{int},R)I_{bc}(s,r_\mathrm{int},R)), \end{aligned} \tag{223}$$

where in the last identity we plugged in relation (210) for $\hat{C}_{ab}(r,\tau,\zeta)$, with (218) substituted in. After the appropriate integrals are performed, it turns out that a reasonable value for x_c is $\pi/2 < x_\mathrm{c} < \pi$. We will, however, not go into the details of either the integration or the approximations.

6.4 Time Dependence

There is another important parameter on which the energy density spectrum for gravitational waves from bubble collisions during phase transitions depends, namely the duration of the phase transition. It is already present in equation (184), where it simply came from the solution to equation (168). For the specific case of bubble collisions during phase transitions however it turns out that the form of $\Pi(k,\tau,\zeta)$ also depends on the duration of the phase transition. It enters quite directly via the function $\phi(t)$ in (216), the fraction of volume occupied by the bubbles, and slightly more indirectly via the bubble radius $R(t)$.

We can express $\phi(t)$ in terms of the probability of its complement, that is, the probability that at given point there has not been a phase transition. This, in turn, we can express in terms of the rougher, thus easier to compute, quantity $I(\eta)$. This quantity is the fraction of volume occupied by bubbles at time η, without considering any overlap. This gives for $\phi(\eta)$ ([4]):

$$\phi(\eta) = 1 - e^{-I(\eta)}. \tag{224}$$

To compute $I(\eta)$ we need the bubble nucleation rate, which is defined as $\Gamma(\eta) = \mathcal{M}^4 a_*^4 e^{-S(\eta)}$, with $\mathcal{M}$ the energy scale of the phase transition, and $S(\eta)$ the tunneling action. Expanding $\Gamma(\eta)$ to first order in η around the time of the end of the phase transition, η_{fin} yields:

$$\Gamma(\eta) \approx \Gamma(\eta_{\text{fin}})(1 + \tilde{\beta}(\eta - \eta_{\text{fin}})) \approx \Gamma(\eta_{\text{fin}}) e^{\tilde{\beta}(\eta - \eta_{\text{fin}})}, \tag{225}$$

with the definition $\tilde{\beta} := -dS/d\eta|_{\eta_{\text{fin}}}$, and using the first order Taylor expansion for the exponential. If we assume a constant velocity v_{b} for the bubble expansion, we see that a bubble formed at time η' occupies a volume of $\frac{4\pi}{3} r^3 = \frac{4\pi}{3} v_b^3 (\eta - \eta')^3$ at time η. The chance for a bubble to form at time η' is given by $\Gamma(\eta')$, so the fraction of volume occupied at time η by bubbles formed at time η' is given by $\frac{4\pi}{3}\Gamma(\eta')v_b^3(\eta-\eta')^3$, not considering overlap. The total volume occupied by bubbles at time η, without considering overlap and under the assumption that the universe remains static throughout the phase transition, is then given by integrating over all times η' between the beginning of the phase transition η_{in} and η:

$$I(\eta) = \frac{4\pi}{3} \int_{\eta_{\text{in}}}^{\eta} d\eta' \Gamma(\eta') v_b^3 (\eta - \eta')^3. \tag{226}$$

Integrating this by parts, denoting by $\Gamma_i(\eta)$ the ith indefinite integral of $\Gamma(\eta)$, gives:

$$I(\eta) = \frac{4\pi v_{\text{b}}}{3} \left(-\int_{\eta_{\text{in}}}^{\eta} d\eta' \Gamma_1(\eta')(-3(\eta - \eta')^2) + \Gamma_1(\eta')(\eta - \eta')^3|_{\eta_{\text{in}}}^{\eta} \right), \tag{227}$$

where we see that the boundary term vanishes, since $\Gamma(\eta) = 0$ for all $\eta \leq \eta_{\text{in}}$, thus $\Gamma_1(\eta_{\text{in}}) = 0$, and $\eta - \eta = 0$. Repeatedly integrating by parts we see that the boundary terms vanish in a similar fashion, until we are left with:

$$I(\eta) = 8\pi v_{\text{b}}^3 \Gamma_4(\eta')|_{\eta_{\text{in}}}^{\eta} \approx 8\pi \frac{v_{\text{b}}^3 \Gamma(\eta_{\text{fin}})}{\tilde{\beta}^4} e^{\tilde{\beta}(\eta - \eta_{fin})} \approx 8\pi \frac{v_{\text{b}}^3}{\tilde{\beta}^4} \Gamma(\eta), \tag{228}$$

where in the penultimate identity we plugged in the approximation from equation (225). We have thus found for $\phi(\eta)$, plugging everything into (224):

$$\phi(\eta) \approx 1 - e^{-8\pi \frac{v_{\text{b}}^3}{\tilde{\beta}}^4 \Gamma(\eta)} \approx 1 - e^{-8\pi \frac{v_{\text{b}}^3}{\tilde{\beta}^4} \Gamma(\eta_{\text{fin}}) \exp(\tilde{\beta}(\eta - \eta_{\text{fin}}))}. \tag{229}$$

We clean this up by noting that because all of space will have undergone the phase transition at η_{fin}, we should have $\phi(\eta_{\text{fin}}) = 1$. We can in principle determine the value of $\Gamma(\eta_{\text{fin}})$ from this. This, however, involves a limiting procedure, which we circumvent by picking a number M such that $\exp(-M) \approx 0$ and defining η_{fin} by $\Gamma(\eta_{\text{fin}}) = \tilde{\beta}^4 M / 8\pi v_b^3$. We do the same for η_{in}, we pick a number m

such that $\exp(-m) \approx 1$ and we define $\eta_{\rm in}$ by $\phi(\eta_{\rm in}) = 1 - \exp(-m)$, that is, $M \exp(\tilde{\beta}(\eta_{\rm in} - \eta_{\rm fin})) = m$. This gives us a relation between the duration of the phase transition and $\tilde{\beta}$:

$$\eta_{\rm fin} - \eta_{\rm in} = \tilde{\beta}^{-1} \ln(\frac{M}{m}). \tag{230}$$

Plugging in what we got so far into (229) this gives for $\phi(\eta)$:

$$\phi(\eta) = 1 - e^{-M \exp(\tilde{\beta}(\eta - \eta_{\rm fin}))}. \tag{231}$$

We have eliminated quite a lot of quantities already, but we still have M and $\tilde{\beta}$ left. It turns out we can eliminate M and give $\tilde{\beta}$ a convenient physical meaning in our discussion of the time dependence of the bubble radius. We thus proceed with trying to find an expression for $R(\eta)$.
To evaluate the bubble radius, we make the simplification of not accounting for the possibility of bubbles having different radii at a given time. This means we are only interested in the mean bubble radius at time η. To find this mean we look at the distribution of the number of bubbles with a given radius δ at η. To find this distribution, consider the total number of bubbles at a given time η, which have the radius up to $\delta = v_b(\eta - \eta_\delta)$, where η_δ is the nucleation time of a bubble that has radius δ at time η, differentiating this cumulative distribution with respect to δ will then yield the number of bubble with radius δ at a given time η. The number of bubbles formed at a given time η' is given by the rate at which bubbles form $\Gamma(\eta')$ times the fraction of space where bubbles still can form $p(\eta') := 1 - \phi(\eta')$, that is, the fraction of space that still has to undergo the phase transition. The number of bubbles $N(\eta)$ with radius up to δ at time η is then given by:

$$N(\eta) = \int_{\eta_\delta}^{\eta} d\eta' \Gamma(\eta') p(\eta'). \tag{232}$$

Differentiating this with respect to the δ at δ' we obtain the distribution of number of bubbles of radius δ' at a given time η:

$$\begin{aligned} \frac{dN}{d\delta}|_\eta &= -\Gamma(\eta_\delta) p(\eta_\delta) \frac{d\eta_\delta}{d\delta}|_{\delta=\delta'} \\ &= -\Gamma(\eta_\delta) p(\eta_\delta) \frac{d}{d\delta}(\eta - \frac{\delta}{v_b})|_{\delta=\delta'} \\ &= \frac{\Gamma(\eta'_\delta) p(\eta'_\delta)}{v_b}, \end{aligned} \tag{233}$$

where in the first identity we used the fundamental theorem of integral calculus, and in the second we plugged in the relation between δ and η_δ stated above. It can be shown (see [16]) that for each η this distribution has a maximum at $\bar{R}(\eta) = \frac{v_{\rm b}}{\beta} \ln I(\eta)$, which we take to be the mean bubble radius at time η, since the distribution looks like a Gaussian[4]. Note that $\ln I(\eta) < 0$ for $I(\eta) < 1$, this would give the unphysical result of a negative bubble radius. To get rid of this, we set $\bar{R}(\eta)$ to zero for all $\eta < \bar{\eta}$, where $\bar{\eta}$ is such that $I(\bar{\eta}) = 1$:

$$\bar{R}(\eta) = \left\{ \begin{array}{ll} 0 & \text{for } \eta_{\rm in} < \eta < \bar{\eta} \\ \frac{v_{\rm b}}{\beta} \ln I(\eta) & \text{for } \bar{\eta} < \eta < \eta_{\rm fin} \end{array} \right\}. \tag{234}$$

[4] The assertion that the maximum and the mean coincide is made in the caption of figure 5 in [16], but no justification is given. The argument given here is an educated guess.

Since in our treatment we assume bubbles to be spherically expanding at constant velocity, we are only evaluating times much later than the bubble nucleation. So we might just as well identify $\eta_{\rm in} \equiv \bar{\eta}$ in our evaluations.
We still want to eliminate M from (231), we can do this by noting that $I(\bar{\eta}) = 1$ implies $1 = M \exp(\beta(\bar{\eta} - \bar{\eta_{\rm fin}}))$, and with the identification $\eta_{\rm in} \equiv \bar{\eta}$ this becomes:

$$\ln M = \tilde{\beta}(\eta_{\rm fin} - \eta_{\rm in}). \tag{235}$$

We can thus rewrite (228) as:

$$\begin{aligned} I(\eta) &= M e^{\tilde{\beta}(\eta - \eta_{\rm fin})} \\ &= e^{\tilde{\beta}(\eta - \eta_{\rm fin}) + \ln M} \\ &= e^{\tilde{\beta}(\eta - \eta_{\rm fin} + \eta_{\rm fin} - \eta_{\rm in})}. \end{aligned} \tag{236}$$

Neglecting the logarithm of $\frac{M}{m}$ from here on[5], we get from (230):

$$\eta_{\rm fin} - \eta_{\rm in} \approx \tilde{\beta}^{-1}, \tag{237}$$

that is, we identify $\tilde{\beta}$ with the duration of the phase transition. We thus get for $I(\eta)$:

$$\begin{aligned} I(\eta) &= e^{\tilde{\beta}(\eta - \eta_{\rm fin} + \tilde{\beta}^{-1})} \\ &= e^{1 + \tilde{\beta}(\eta - \eta_{\rm fin})}. \end{aligned} \tag{238}$$

and $\phi(\eta)$ becomes, plugging in equation (238) for $I(\eta)$:

$$\phi(\eta) = 1 + e^{-\exp(1 + \beta(\eta - \eta_{\rm fin}))}. \tag{239}$$

Plugging in (238) into (234), along with identification the $\eta_{\rm in} \equiv \bar{\eta}$, we get for $\eta_{\rm in} < \eta < \eta_{\rm fin}$:

$$\begin{aligned} \bar{R}(\eta) &= \frac{v_{\rm b}}{\tilde{\beta}}(\ln(e) + \ln(e^{\tilde{\beta}(\eta - \eta_{\rm fin})})) \\ &= v_{\rm b}(\tilde{\beta}^{-1} + \eta - \eta_{\rm fin}) \\ &\approx v_{\rm b}(\eta - \eta_{\rm in}), \end{aligned} \tag{240}$$

where in the last step we used (237). We have now computed the quantities necessary to calculate the energy density spectrum.

6.5 Energy-Density Spectrum Today

We can now use equation (197) to compute the energy density spectrum today. Deciding not to care about the rain forest any more, we will collect the quantities from equations (222) and (239) and plug them into expression (192) for S_k. Before we do this, however, we will have a look at what simplifications we can make. If we look at $w(\tau)$, the enthalpy density, we notice that this quantity

[5] No justification for this effectively taking $\ln M/m \approx 1$ is given in either [4] or [16], although a remark on how to include the factor is made in [4]. However, looking at what M and m are, we see that $M \gg m$, so since both are positive $M/m \gg 1$. This leads to the conclusion $\ln(M/m) \gg 1$. The only reason I see for this approximation is the $\ln(M)$ on the left hand side of (235).

evolves at scales of one Hubble time, much longer than we assume the phase transition to last. We thus take it to be constant with value

$$w_* = \frac{4}{3}\rho^*_{\rm rad} \tag{241}$$

during the phase transition, where $\rho^*_{\rm rad}$ is the energy density at the time of the phase transition. Another simplification comes from the form of the energy-momentum tensor (199). Assuming radiation dominance we get for the kinetic part of the energy-momentum tensor:

$$T^{\rm kin}_{\mu\nu}(\tau) = \frac{4}{3}\rho_{\rm rad}(\tau)U_\mu(\tau)U_\nu(\tau), \tag{242}$$

so, plugging in $U_0(\tau) = \gamma(\tau)v(\tau)$, we get:

$$T^{\rm kin}_{00}(\tau) = \rho_{\rm kin}(\tau) = \frac{4}{3}\rho_{\rm rad}(\tau)\frac{v(\tau)^2}{1-v(\tau)^2} \tag{243}$$

and thus for the ratio of kinetic energy and radiation energy at the time of the phase transition:

$$\frac{\Omega^*_{\rm kin}}{\Omega^*_{\rm rad}} = \frac{\rho^*_{\rm kin}}{\rho^*_{\rm rad}} = \frac{4}{3}\frac{(\frac{r_{\rm int}}{R}v_{\rm f})^2}{1-(\frac{r_{\rm int}}{R}v_{\rm f})^2}. \tag{244}$$

We now make the assumption that the velocity is constant, and has value equal to the fluid velocity at the inner boundary $\frac{r_{\rm int}}{R}v_{\rm f}$. From the hydrodynamics follows that this velocity is always (also for the case of deflagrations) strictly less than the speed of sound, see [4], which is $1/\sqrt{3}$ for relativistic fluids. This ensures that we always have $\Omega^*_{\rm kin}/\Omega^*_{\rm rad} < 1$ which has to be satisfied in order for the universe to be isotropic, and this shows we can safely apply this assumption everywhere. We thus get for S_k:

$$\begin{aligned} S_k &= \frac{Gk^3a_*^2}{32\pi^7}\left(\frac{w_*v_f^2}{1-v_{\rm in}^2}\right)^2 \int_{\tau_i}^{\tau_f} d\tau \int_{\tau_i}^{\tau_f} d\zeta \int d^3\mathbf{p} \int d^3\mathbf{r} \int d^3\mathbf{s} \\ &\times \cos(k\tau - k\zeta)\left(\frac{1+e^{-\exp(1+\tilde{\beta}(\eta-\eta_{\rm fin}))}}{V_{\rm c}R(\tau)^2}\right)^2 P_{abcd}e^{i\mathbf{r}\cdot(\mathbf{k}-\mathbf{p})}e^{i\mathbf{s}\cdot\mathbf{p}} \\ &\times ((I_{ac}(r,r_{\rm int},R)I_{bd}(s,r_{\rm int},R) + I_{ad}(r,r_{\rm int},R)I_{bc}(s,r_{\rm int},R)) \\ &\times \Theta(\zeta-\tau)\Theta(\frac{x_{\rm c}}{k} - (\zeta-\tau)) + \text{symmetrized}(\tau,\zeta)). \end{aligned} \tag{245}$$

Deciding to care for the rain forest again, we consider only the pre-factor in front of the integral in evaluating (197), where we have set $a_f = a_*$:

$$\begin{aligned} \frac{h^2}{\rho_{\rm c}}\left(\frac{g_*}{g_0}\right)^{-\frac{1}{3}}\frac{\rho_{\rm rad0}}{\rho^*_{\rm rad}}\frac{Gk^3}{32\pi^7a_*^2}\left(\frac{w_*v_f^2}{1-v_{\rm in}^2}\right)^2 &= \left(\frac{g_0}{g_*}\right)^{\frac{1}{3}}\frac{3h^2\Omega^0_{\rm rad}k^3}{256\pi^8a_*^4}\frac{8\pi Ga_*^2\rho^*_{\rm rad}}{3} \\ &\times \left(\frac{\Omega^*_{\rm kin}}{\Omega^*_{\rm rad}}\right)^2\left(\frac{R}{r_{\rm int}}\right)^4, \end{aligned} \tag{246}$$

where we have used (241) for w_* and used (244) for the velocity quotient. From equation (88) we have:

$$H_*^2 = \frac{8\pi G}{3}\rho^* = \frac{8\pi G}{3}\rho^*_{\rm rad}, \tag{247}$$

with in the last identity the assumption of radiation dominance plugged in again. Defining the conformal Hubble factor as $\mathcal{H}_* = a_* H_*$ we can rewrite (246) as [6]:

$$\left(\frac{g_0}{g_*}\right)^{\frac{1}{3}} \frac{3h^2\Omega^0_{\text{rad}}\mathcal{H}_*^2 k^3}{256\pi^8 a_*^4} \left(\frac{\Omega^*_{\text{kin}}}{\Omega^*_{\text{rad}}}\right)^2 \left(\frac{R}{r_{\text{int}}}\right)^4 . \tag{248}$$

As said before, we will not go into the details of the integration of expression (245). There are, however, important results to be obtained from it. Looking at the expression (239) for $\phi(\eta)$, we see that integrating the corresponding factor $\phi(\eta)$ in (245) will give us a factor $1/\tilde{\beta}^2$. This can be made explicit by performing a suitable variable substitution (see [4]). Combining this with the result (248) for the pre-factor, we see that the abundance of gravitational wave energy density today scales as $\left(\frac{\mathcal{H}_*}{\tilde{\beta}}\right)^2 = \left(\frac{\eta_{\text{fin}}-\eta_{\text{in}}}{\mathcal{T}}\right)^2$, the square of the ratio between the lenght of the phase transition and the Hubble time $\mathcal{T}$. So the longer the phase transition takes, the larger the abundance. Also, we see that the spectrum has an overall k^3 scaling, but there is still k dependence hidden in the rest of the expression for the spectrum. Taking these factors into account as is done in the appendix of [4] a k^{-2} scaling is found at higher frequencies. As mentioned before, gravitational waves from bubble collisions is a field of ongoing research, and one should note that different power laws can be found in different research articles on the subject, like for example in [12] a k^{-1} scaling is found for higher frequencies in the simulations done there. In [3] the authors of [4, 12] have worked together to combine the merits of their respective methods, and to correct some flaws in the analytic model presented here.
Finally, the result found for the spectrum of abundance of gravitational wave energy density is:

$$\begin{aligned} h^2 \frac{d\Omega_{\text{gw}}}{d\ln(k)} &= \left(\frac{g_0}{g_*}\right)^{\frac{1}{3}} \frac{3h^2\Omega^0_{\text{rad}}\mathcal{H}_*^2 k^3}{256\pi^8 a_*^4} \left(\frac{\Omega^*_{\text{kin}}}{\Omega^*_{\text{rad}}}\right)^2 \left(\frac{R}{r_{\text{int}}}\right)^4 \\ &\times \int_{\tau_i}^{\tau_f} d\tau \int_{\tau_i}^{\tau_f} d\zeta \int d^3\mathbf{p} \int d^3\mathbf{r} \int d^3\mathbf{s} \cos(k\tau - k\zeta) \\ &\times \left(\frac{1 + e^{-\exp(1+\tilde{\beta}(\eta-\eta_{\text{fin}}))}}{V_{\text{c}} R(\tau)^2}\right)^2 P_{abcd} e^{i\mathbf{r}\cdot(\mathbf{k}-\mathbf{p})} e^{i\mathbf{s}\cdot\mathbf{p}} \\ &\times ((I_{ac}(r, r_{\text{int}}, R) I_{bd}(s, r_{\text{int}}, R) I_{ad}(r, r_{\text{int}}, R) I_{bc}(s, r_{\text{int}}, R)) \\ &\times \Theta(\zeta - \tau)\Theta(\frac{x_{\text{c}}}{k} - (\zeta - \tau)) \\ &+ \text{symmetrized}(\tau, \zeta)) \end{aligned} \tag{249}$$

Upon integration, one can extract the form of the spectrum from this, as shown

[6]Notice that we find we have an apparent $a^4/(2^7\pi^5 a_*^4)$ discrepancy between our result and that in formula (57) from [4]. Looking at formula (38) there, we see that a factor of $1/(4\pi)^2$ is due to not having performed the isotropic spatial integrals yet. The factor a^4 is only there because (57) in [4] is valid for all $\eta > \eta_*$, whereas (248) here is already evaluated for today. When comparing, we should evaluate (57) today, using the convention $a_0 = 1$ from [4], and we see the a^4 is set to 1. The factor $1/a_*^4$ can be traced back to a rather strange difference in the equations of motions used here, (168), and in [4], (9). Looking at (8) in [4], one expects that our $T^{(\text{TT})}_{ij}(\mathbf{k}, \eta)$ coincides with their $\Pi_{ij}(\mathbf{k}, \eta)$, but this is in contradiction with the a_*^2 on the right hand side of (9) from [4] (where we have already taken into account the difference between $\tilde{h}_{ij}$ here and h_{ij} there). We are then left with a discrepancy of $1/8\pi^3$, which may very well be due to a factor $(2\pi)^3$ one should add in the definition (209) when using our Fourier-convention.

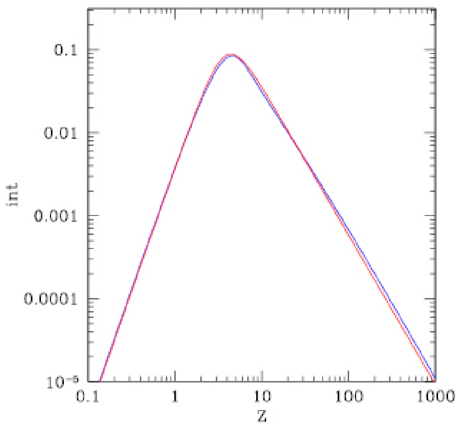

Figure 5: The shape (without pre-factors) of gravitational wave energy density spectrum as a function of $Z = kv_{out}/\tilde{\beta}$. For this plot $x_c = 0.9\pi$. Plotted is the actual shape as found by integrating and by good a approximation. The figure is from [4].

in figure 5. It turns out that the shape of the spectrum does not depend much on the strength of the phase transition, which can be expressed in terms of $\alpha = \rho_{\rm vac}/\rho^*_{\rm rad}$ where $\rho_{\rm vac}$ is the energy density of the false vacuum and $\rho^*_{\rm rad}$, the significant dependence on the strength of the phase transition is in the $(\Omega^*_{\rm kin}/\Omega^*_{\rm rad})^2$ factor, and implicitly in the factor $\frac{R}{r_{\rm int}}$. The integral also contains a dependence on the latter factor, we can factorize this out as $(1 - (\frac{r_{\rm int}}{R})^3)^2$. The dependence of the spectrum on the strength of the phase transition is therefore almost merely scaling. One important feature of this spectrum is its peak frequency $f_{\rm peak}$ and the associated energy density. It turns out that at time of emission we have $f_{\rm peak} \approx \tilde{\beta}/v_{\rm out}$, the frequency associated with the largest bubble size that can be obtained, $(\eta_{\rm fin} - \eta_{\rm in})v_{\rm out}$. Translated to the physical frequency today, this becomes:

$$f_{\rm peak} \simeq 1.12 \times 10^{-2}{\rm mHz} \left(\frac{g_*}{100}\right)^{\frac{1}{6}} \frac{T_*}{100{\rm GeV}} \frac{\tilde{\beta}}{\mathcal{H}_*} \frac{1}{v_{\rm out}}, \tag{250}$$

where T_* denotes the temperature of the universe at the time of the phase transition. Evaluating this for a typical first order electroweak phase transition with $\tilde{\beta}/\mathcal{H}_* = 100$, $T_* = 100$GeV and $g_* \sim 100$ leads to $f_{\rm peak} \sim 1{\rm mHz}/v_{\rm out}$, constraining the frequency to lie above $f_{\rm peak} \sim 1$mHz. We can plot the height of the peak in the spectrum as a function of α and $v_{\rm b}$ for different values for the duration of the phase transition, expressed in terms of $\beta/H_* = \tilde{\beta}/\mathcal{H}_*$, this is shown in figure 6.

6.6 Likelihood of Detection

The reason physicists are interested in gravitational waves from bubble collisions during first-order phase transitions is that one of the most important phase transitions that is thought to have occurred in our early universe, the electroweak phase transition, could be a first-order phase transition. We have seen from

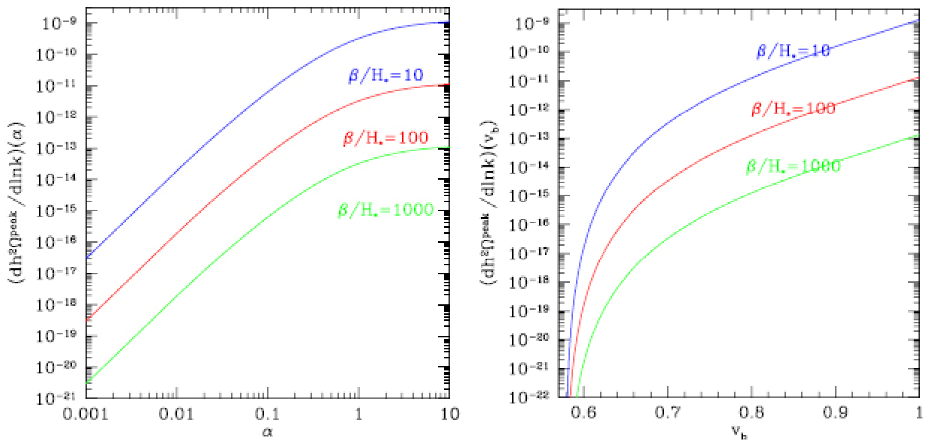

Figure 6: The left panel shows the peak in the spectrum as a function of α, the right as a function of v_{b}. Figure from [4].

the discussion of the pre-factor in (248) that there is quite a lot of information about the phase transition contained in the energy density spectrum. It would therefore be really exiting if these gravitational waves could be detected. The detector planned up to now that is most likely to be able to detect these gravitational waves is the upcoming LISA. At the best sensitivity this detector should be able to detect energy-densities almost as small as $\Omega h^2 \sim 10^{-12}$. Using the model developed in this chapter, one can predict (see [4]) that one would need a fluid velocity of around $0.2c$ to generate this energy density when the phase transition lasts around one tenth of a Hubble time. For the more realistic case of a duration of one hundredth of a Hubble time, one would need the violent speed of $0.5c$. Based on this, it would seem somewhat unlikely LISA will see gravitational waves generated by bubble collisions during the electroweak phase transition. Frequency wise LISA listens to a range from $f \sim 3 \times 10^{-2}$mHz to $f \sim 10^{-1}$Hz ([1]), with the peak sensitivity lying at $f \sim 2$mHz ([4]). So for high velocities the peak frequency will be close to the peak sensitivity of LISA. The authors of [4] have promised to do a publication on the likelihood of detection of these gravitational wave signals from the electroweak phase transition, which may present more optimistic predictions for the energy density spectrum. Of course, physicists are already working on another bigger detector, which will be even more sensitive than the already pretty advanced LISA. This detector will be called Big Bang Observatory (BBO), and will be a bigger version of LISA, also launched by NASA. This detector will obviously have a better chance of detecting the gravitational waves from bubble collisions during the electroweak phase transition.

7 Conclusion

In this thesis we have given an introduction to the general theory of gravitational waves. We have derived the equations of motion for gravitational waves (33) on Minkowski space, and given some intuition for what gravitational waves do with space-time, and what kind of phenomena generate them. With this intuition in hand we derived the equations of motion for gravitational waves in a FLRW-universe (132).
We proceeded with the more advanced topic of finding the energy-density spectrum for gravitational waves. We started out general, and then restricted ourselves to the case of generation during radiation domination in a FLRW-universe. We derived formula (197) for the abundance of gravitational wave energy-density today, and discussed various approaches and simplifications along the way.
Finally, we applied what we found so far to the case of gravitational waves from bubble-collisions during first-order phase transitions. There we described an analytic model for the bubble collisions, which incorporated the assumptions that: the phase transition takes place during radiation domination, the velocity inside a bubble increases linearly with the distance from the center, we can account for colliding bubbles by considering correlators of overlapping bubbles, Wick's theorem can be used to approximate four-point velocity correlators (even though the velocities do not have a Gaussian distribution), all bubbles have the same radius, the unequal time correlator for the tenser anisotropic stress in Fourier space can be approximated by its value for equal times with some correction, and $\rho + p$ and γ do not depend on the position. Under these assumptions we found (249) for the spectrum of abundance, where we left the integrals unperformed, since this would be to technical to treat in a text on this level. For a treatment, see [4]. In spite of not having performed the integrals, we could still extract physical meaning from the pre-factor (248). Our main conclusions there were that the longer the phase transition, the larger the abundance, and more violent the phase transition, the larger the abundance. We further discussed, without performing the integration ourselves, different properties of the spectrum, such as the peak frequency and the k scaling.

Acknowledgments

I would like to thank Renee Hoekzema for her support, ranging from moral support via just listening to me finding my own mistakes to valuable advice. I would also like to thank my supervisor Tomislav Prokopec, who showed me that theoretical physics research articles are not always as scary and horribly complicated as they look.

Appendix A

The components of the perturbation to the Ricci-tensor are given by:

$$\delta R_{\mu\nu} = \partial_\lambda \delta\Gamma^\lambda_{\mu\nu} - \partial_\nu \delta\Gamma^\lambda_{\mu\lambda} + \delta\Gamma^\lambda_{\kappa\lambda}\bar{\Gamma}^\kappa_{\mu\nu} + \delta\Gamma^\kappa_{\mu\nu}\bar{\Gamma}^\lambda_{\kappa\lambda} - \delta\Gamma^\kappa_{\mu\lambda}\bar{\Gamma}^\lambda_{\kappa\nu} - \delta\Gamma^\lambda_{\kappa\nu}\bar{\Gamma}^\kappa_{\mu\lambda} \tag{251}$$

In calculating this we will frequently need:

$$\begin{aligned}
\delta\Gamma^\lambda_{\lambda 0} &= -\frac{1}{2}\partial_0 h_{00} + \frac{1}{2a^2}(-\frac{2\dot{a}}{a}h_{ii} + \partial_0 h_{ii} + \partial_i h_{0i} - \partial_i h_{0i}) \\
&= -\frac{1}{2}\partial_0 h_{00} - \frac{\dot{a}}{a^3}h_{ii} + \frac{1}{2a^2}\partial_0 h_{ii} \\
&= \partial_0(\frac{1}{2a^2}h_{ii} - \frac{1}{2}h_{00})
\end{aligned}$$

and

$$\begin{aligned}
\delta\Gamma^\lambda_{\lambda i} &= \frac{\dot{a}}{a}h_{i0} - \frac{1}{2}\partial_i h_{00} + \frac{1}{2a^2}(-2a\dot{a}h_{j0}\delta_{ij} + \partial_i h_{jj} + \partial_j h_{ij} - \partial_j h_{ij}) \\
&= -\frac{1}{2}\partial_i h_{00} + \frac{1}{2a^2}\partial_i h_{jj} \\
&= \partial_i(\frac{1}{2a^2}h_{jj} - \frac{1}{2}h_{00}),
\end{aligned}$$

which can be combined into the more compact form:

$$\delta\Gamma^\lambda_{\lambda\mu} = \partial_\mu(\frac{1}{2a^2}h_{jj} - \frac{1}{2}h_{00}) \tag{252}$$

We then get for the components of the perturbation of the Ricci tensor (equation 101), using $\partial_0 h_{\mu\nu} = \dot{h}_{\mu\nu}$ for shortness of notation:

$$\begin{aligned}
\delta R_{ij} &= -\frac{1}{2}\partial_0(-2a\dot{a}h_{00}\delta_{ij} + \partial_j h_{0i} + \partial_i h_{0j} - \partial_0 h_{ij}) \\
&+ \partial_k\left(\frac{1}{2a^2}(-2a\dot{a}h_{k0}\delta_{ij} + \partial_i h_{jk} + \partial_j h_{ij} - \partial_j h_{ij})\right) \\
&- \partial_j\partial_i(\frac{1}{2a^2}h_{kk} - \frac{1}{2}h_{00}) + \partial_0(\frac{1}{2a^2}h_{kk} - \frac{1}{2}h_{00})a\dot{a}\delta_{ij} \\
&- \frac{1}{2}(-2a\dot{a}h_{00}\delta_{ij} + \partial_j h_{0i} + \partial_i h_{0j} - \dot{h}_{ij})\frac{\dot{a}}{a}\delta_{kk} \\
&+ \frac{1}{2}(-2a\dot{a}h_{00}\delta_{ik} + \partial_k h_{0i} + \partial_i h_{0k} - \dot{h}_{ik})\frac{\dot{a}}{a}\delta^k_j \\
&- \frac{1}{2a^2}(-\frac{2\dot{a}}{a}h_{ki} + \dot{h}_{ki} + \partial_i h_{0k} - \partial_k h_{0i})a\dot{a}\delta_{kj} \\
&+ \frac{1}{2}(-2a\dot{a}h_{00}\delta_{jk} + \partial_k h_{0j} + \partial_j h_{0k} - \dot{h}_{jk})\frac{\dot{a}}{a}\delta^k_i \\
&- \frac{1}{2a^2}(-\frac{2\dot{a}}{a}h_{kj} + \dot{h}_{kj} + \partial_j h_{0k} - \partial_k h_{0j})a\dot{a}\delta_{ki} \\
&= (\dot{a}^2 + a\ddot{a})h_{00}\delta_{ij} + a\dot{a}\dot{h}_{00}\delta_{ij} - \frac{1}{2}(\partial_j\dot{h}_{0i} + \partial_i\dot{h}_{0j} - \ddot{h}_{ij}) - \frac{\dot{a}}{a}\partial_k h_{k0}\delta_{ij} \\
&+ \frac{1}{2a^2}(\partial_k\partial_i h_{jk} + \partial_k\partial_j h_{ij} - \nabla^2 h_{ij} - \partial_j\partial_i h_{kk}) + \frac{1}{2}\partial_j\partial_i h_{00} \\
&- \frac{\dot{a}^2}{2a^2}h_{kk}\delta_{ij} + \frac{\dot{a}}{2a}\dot{h}_{kk}\delta_{ij} - \frac{1}{2}a\dot{a}\dot{h}_{00}\delta_{ij} + 3\dot{a}^2 h_{00}\delta_{ij} \\
&- \frac{3\dot{a}}{2a}(\partial_j h_{0i} + \partial_i h_{0j} - \partial_0 h_{ij}) - \dot{a}^2 h_{00}\delta_{ij} + \frac{\dot{a}}{2a}(\partial_j h_{0i} + \partial_i h_{0j} - \partial_0 h_{ij}) \\
&+ \frac{\dot{a}^2}{a^2}h_{ij} - \frac{\dot{a}}{2a}(\partial_0 h_{ij} + \partial_i h_{0j} - \partial_j h_{0i}) - \dot{a}^2 h_{00}\delta_{ij} \\
&+ \frac{\dot{a}}{2a}(\partial_i h_{0j} + \partial_j h_{0i} - \partial_0 h_{ij}) + \frac{\dot{a}^2}{a^2}h_{ij} - \frac{\dot{a}}{2a}(\partial_0 h_{ij} + \partial_j h_{0i} - \partial_i h_{0j}) \\
&= \frac{1}{2}\partial_j\partial_i h_{00} + (\dot{a}^2 + a\ddot{a})h_{00}\delta_{ij} + \frac{1}{2}a\dot{a}\dot{h}_{00}\delta_{ij} + \frac{\dot{a}}{2a}(\dot{h}_{kk}\delta_{ij} - \dot{h}_{ij}) + \frac{1}{2}\ddot{h}_{ij} \\
&+ \frac{1}{2a^2}(\partial_k\partial_i h_{jk} + \partial_k\partial_j h_{ij} - \nabla^2 h_{ij} - \partial_j\partial_i h_{kk}) + \frac{\dot{a}^2}{a^2}(-h_{kk}\delta_{ij} + 2h_{ij}) \\
&- \frac{\dot{a}}{a}\partial_k h_{k0}\delta_{ij} - \frac{1}{2}(\partial_j\dot{h}_{0i} + \partial_i\dot{h}_{0j}) + \dot{a}^2 h_{00}\delta_{ij} - \frac{3\dot{a}}{2a}(\partial_j h_{0i} + \partial_i h_{0j}) \\
&- \frac{\dot{a}}{2a}(\partial_j h_{0i} + \partial_i h_{0j})
\end{aligned} \tag{253}$$

$$
\begin{aligned}
\delta R_{0j} &= \delta R_{j0} = \partial_0(\frac{\dot a}{a}h_{j0} - \frac{1}{2}\partial_j h_{00}) + \partial_i \frac{1}{2a^2}(-\frac{2\dot a}{a}h_{ij} + \dot h_{ij} + \partial_j h_{i0} - \partial_i h_{j0}) \\
&- \partial_j\partial_0(\frac{1}{2a^2}h_{ii} - \frac{1}{2}h_{00}) + \partial_i(\frac{1}{2a^2}h_{kk} - \frac{1}{2}h_0 0)\frac{\dot a}{a}\delta^i_j \\
&+ (\frac{\dot a}{a}h_{j0} - \frac{1}{2}\partial_j h_{00})\frac{\dot a}{a}\delta^k_k - \frac{1}{2a^2}(2\dot h_{i0} - \partial_i h_{00})a\dot a\delta_{ij} - (\frac{\dot a}{a}h_{i0} \\
&- \frac{1}{2}\partial_i h_{00})\frac{\dot a}{a}\delta^i_j \\
&- \frac{1}{2a^2}(-2a\dot a h_{i0}\delta_{jk} + \partial_k h_{ij} + \partial_j h_{ik} - \partial_i h_{jk})\frac{\dot a}{a}\delta^k_i \\
&= (\frac{\ddot a}{a} - \frac{\dot a^2}{a^2})h_{0j} + \frac{\dot a}{a}\dot h_{j0} - \frac{1}{2}\partial_j \dot h_{00} - \frac{\dot a}{a^3}\partial_i h_{ij} \\
&+ \frac{1}{2a^2}(\partial_i \dot h_{ij} + \partial_i\partial_j h_{i0} - \nabla^2 h_{j0}) \\
&+ \frac{\dot a}{a^3}\partial_j h_{ii} - \frac{1}{2a^2}\partial_j \dot h_{ii} + \frac{1}{2}\partial_j \dot h_{00} + \frac{\dot a}{2a^3}\partial_j h_{kk} - \frac{\dot a}{2a}\partial_j h_{00} + 3\frac{\dot a^2}{a^2}h_{j0} \\
&- \frac{3\dot a}{2a}\partial_j h_{00} - \frac{\dot a}{a}\dot h_{0j} + \frac{\dot a}{2a}\partial_j h_{00} - \frac{\dot a^2}{a^2}h_{0j} + \frac{\dot a}{2a}\partial_j h_{00} + \frac{\dot a^2}{a^2}h_{j0} - \frac{\dot a}{2a^3}\partial_j h_{ii} \\
&= -\frac{\dot a}{a}\partial_j h_{00} + \frac{1}{2a^2}(\partial_i\partial_j h_{i0} - \nabla^2 h_{j0}) + (\frac{\ddot a}{a} + \frac{2\dot a^2}{a^2})h_{0j} \\
&- (\frac{1}{2a^2}\partial_j \dot h_{ii} - \partial_i \dot h_{ij}) + \frac{\dot a}{a^3}(\partial_j h_{ii} - \partial_i h_{ij}) \\
&= -\frac{\dot a}{a}\partial_j h_{00} + \frac{1}{2a^2}(\partial_i\partial_j h_{i0} - \nabla^2 h_{j0}) + (\frac{\ddot a}{a} + \frac{2\dot a^2}{a^2})h_{0j} \\
&- \frac{1}{2}\partial_0(\frac{1}{a^2}(\partial_j h_{ii} - \partial_i h_{ij}))
\end{aligned}
\tag{254}
$$

$$
\begin{aligned}
\delta R_{00} &= -\frac{1}{2}\ddot h_{00} + \frac{1}{2a^2}\partial_i(2\dot h_{0i} - \partial_i h_{00}) - \partial_0^2(\frac{1}{2a^2}h_{ii} - \frac{1}{2}h_{00}) - \frac{1}{2}\dot h_{00}\frac{\dot a}{a}\delta^i_i \\
&- \frac{1}{a^2}(-\frac{2\dot a}{a}h_{ij} + \dot h_{ij} + \partial_j h_{i0} - \partial_i h_{0j})\frac{\dot a}{a}\delta_{ij} \\
&= \frac{1}{a^2}\partial_i \dot h_{0i} - \frac{1}{2a^2}\nabla^2 h_{00} - \frac{1}{2a^2}\ddot h_{ii} - (3\frac{\dot a^2}{a^4} - \frac{\ddot a}{a^3})h_{ii} \\
&- \frac{3\dot a}{2a}\dot h_{00} + \frac{2\dot{a}^2}{a^4}h_{ii} - \frac{\dot a}{a^3}\dot h_{ii} \\
&= -\frac{1}{2a}\nabla^2 h_{00} - \frac{3\dot a}{2a}\dot h_{00} + \frac{1}{a^2}\partial_i \dot h_{0i} \\
&- \frac{1}{2a^2}(\ddot h_{ii} - \frac{2\dot a}{a}\dot h_{ii} + 2(\frac{\dot a^2}{a^2} - \frac{\ddot a}{a})h_{ii})
\end{aligned}
\tag{255}
$$

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